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On construction of strong orthogonal arrays and column-orthogonal strong orthogonal arrays of strength two plus

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Abstract Computer experiments require space-filling designs with good low-dimensional projection properties. Strong orthogonal arrays are a type of space-filling design that provides better stratifications in low dimensions than ordinary orthogonal arrays. In this paper, we address the problem of constructing strong orthogonal arrays and column-orthogonal strong orthogonal arrays of strength two plus. Existing methods typically rely on regular designs or specific nonregular designs as base orthogonal arrays, limiting the sizes of the final designs. Instead, we propose two general methods that are easy to implement and applicable to a wide range of base orthogonal arrays. These methods produce space-filling designs that can accommodate a large number of factors, provide significant flexibility in terms of run sizes, and possess appealing low-dimensional projection properties. Therefore, these designs are ideal for computer experiments.

Keywords computer experiment, difference scheme, strong orthogonal array, column-orthogonality, space-filling property

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1 Introduction

Computer experiments often involve a large number of factors, but only a few of them (which are unknown before the experiment) may be active, according to the effect sparsity principle. Therefore, computer experiments call for space-filling designs that have good projections to all the low-dimensional subspaces of the factors [6, 21]. One popular method of obtaining such designs is through orthogonal arrays, which offer guaranteed space-filling properties in low-dimensional projections. McKay et al. [16] initiated this line of research by introducing Latin hypercubes, which are essentially orthogonal arrays of strength one

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and provide the maximum uniformity in all univariate projections. Owen [18] and Tang [29] extended this idea by proposing orthogonal array-based designs that achieve stratifications in t-dimensional projections if an orthogonal array of strength t is employed.

He and Tang [9] introduced strong orthogonal arrays, also known as stratum orthogonal arrays [7], which are a new type of design that can achieve finer stratifications than ordinary orthogonal arrays of the same strength. Since then, strong orthogonal arrays have attracted considerable interest from researchers [8, 10, 15, 25, 31, 32], studying their characterizations and new types. Besides their stratification benefits, strong orthogonal arrays also perform well under other space-filling criteria [3, 27, 30] and have various applications, including the optimization of braking performance for freight trains [17] and hyperparameter tuning in deep neural networks [24].

Among the various types of strong orthogonal arrays, those of strength 2+ proposed by He et al. [8] are highly practical. These arrays possess the same two-dimensional space-filling property as strength-three strong orthogonal arrays but require significantly fewer runs. He et al. [8] used regular designs to construct these arrays, which limited their run sizes to prime powers. Nonregular designs have also been employed for constructing such arrays [4, 5, 13], but their methods lack generality as they rely on specific orthogonal arrays or computer searches. In computer experiments, column-orthogonality is an important design criterion since it enables uncorrelated estimations of linear main effects of factors and facilitates space-filling designs under Gaussian process modeling [1]. Zhou and Tang [32] developed strong orthogonal arrays of strength 2+ with column-orthogonality, allowing for flexible run sizes. However, these designs often have a limited number of factors when the number of levels exceeds four (refer to Table 5 in Section 4).

In this paper, we employ a broad class of orthogonal arrays as base orthogonal arrays, which are generated by small regular orthogonal arrays and difference schemes. Leveraging these orthogonal arrays, we introduce two general methods for constructing strength-2+ strong orthogonal arrays. Our methods are capable of generating numerous new strong orthogonal arrays and column-orthogonal strong orthogonal arrays. The proposed constructions encompass the run sizes of designs previously developed through existing methods for cases where the number of levels exceeds four, while significantly increasing the number of factors. Moreover, we demonstrate that under mild conditions, the constructed strong orthogonal arrays are complete, meaning that no additional columns can be added to maintain strength 2+. Simultaneously, we propose using the three-dimensional projection properties to further compare and screen the two types of strength-2+ designs and introduce two useful theorems that can assist in achieving this.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and background and give two examples of strong orthogonal arrays constructed from our methods. The general methods for constructing strong orthogonal arrays and column-orthogonal strong orthogonal arrays of strength 2+ are presented in Sections 3 and 4, respectively. In Section 5, we investigate the three-dimensional projections of the constructed strong orthogonal arrays. Finally, we conclude the paper with a discussion of our findings in Section 6. All the proofs and some small difference schemes used in our constructions can be found in the appendices.

2 Preliminaries and examples

2.1 Notations and definitions

An orthogonal array of N runs, m factors and strength t, denoted by $\mathrm{OA}(N, m, s_1 \times \cdots \times s_m, t)$, is an $N \times m$ matrix whose entries are taken from a set of s_j elements in the j-th column, such that all possible combinations appear with the same frequency in any of its $N \times t$ submatrices. If $s_1 = \cdots = s_m = s$, the array is called symmetrical, and a simpler notation $\mathrm{OA}(N, m, s, t)$ or $\mathrm{OA}(t)$ is used; otherwise it is called asymmetrical. A necessary condition for an $\mathrm{OA}(N, m, s, 2)$ to exist is $N - 1 \geqslant m(s-1)$, and the array is said to be saturated if the equality holds. [12] is an excellent general reference for orthogonal arrays.

Let $\mathrm{GF}(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}$ denote a Galois field of order s, where s is a prime power, $\alpha_0 = 0$,

$\alpha_1 = \alpha$ is a primitive element of GF(s), and $\alpha_2 = \alpha^2, \ldots, \alpha_{s-1} = \alpha^{s-1} = 1$. When an OA($N, m, s, t$) has its entries from GF(s), it is called a regular design if its rows form a linear subspace of the full factorial s^m design over GF(s) or a coset. A saturated regular design can be generated by the Rao-Hamming construction [12, Subsection 3.4], which works as follows. Let $l_1, \ldots, l_r$ where $r = (s^n - 1)/(s - 1)$ be all the vectors in $[\mathrm{GF}(s)]^n$ whose first nonzero element is 1. Take all the linear combinations of the rows of $L = (l_1, \ldots, l_r)$ to obtain a matrix $A = (a_1, \ldots, a_r)$. Then A is an OA($s, 1, s, 1$) if $n = 1$, and a saturated OA($s^n, r, s, 2$) if $n \geqslant 2$. Usually, L is called the generator matrix of A, and l_i is used as the column label of a_i for $i = 1, \ldots, r$.

An $N \times m$ matrix with entries from an abelian group G is called a difference scheme, and is denoted by D(N, m, s), if each element of G appears equally often in the difference vector between any two columns of the matrix [2]. Clearly, we must have $N = \lambda s$ for some positive integer λ. For two matrices $A = (a_{ij})_{N_1 \times m_1}$ and $B = (b_{ij})_{N_2 \times m_2}$ with entries from G, the Kronecker sum of A and B, denoted by $A \oplus B$, is the $N_1 N_2 \times m_1 m_2$ matrix $(a_{ij} + B)$, where $a_{ij} + B$ stands for the $N_2 \times m_2$ matrix with entries $a_{ij} + b_{kl}$ ($1 \leqslant k \leqslant N_2$, $1 \leqslant l \leqslant m_2$). If A is a D(N_1, m_1, s) and B is a D(N_2, m_2, s), both based on an abelian group G, $A \oplus B$ is a D($N_1 N_2, m_1 m_2, s$) [26].

An orthogonal array can be constructed by performing the Kronecker sum operation on a small orthogonal array and a difference scheme [2], and the resolvability properties of such an array enable the addition of more columns [12, Subsection 6.2]. We summarize this fact into the following lemma.

Lemma 2.1. *Suppose that A_0 is an* OA(N, m, s, t_1) *with $m = t_1 = 1$ or $m \geqslant t_1 \geqslant 2$, D_0 is a* D($\lambda s, c, s$)*, and H is an* OA($\lambda s, k, s, t_2$) *with $k = t_2 = 1$ or $k \geqslant t_2 \geqslant 2$, all with entries from an abelian group G. Then*

$$D_1 = (A_0 \oplus D_0, 0_N \oplus H) \tag{2.1}$$

is an OA($\lambda s N, mc + k, s, 2$)*, where 0_N is an N-vector of zeros.*

For a positive integer s, let $\mathbb{Z}_s = \{0, 1, \ldots, s-1\}$. An $N \times m$ matrix with entries from $\mathbb{Z}_{s^2}$ is called a strong orthogonal array of strength 2+ with N runs, m factors and s^2 levels, and is denoted by SOA($N, m, s^2, 2+$), if any two-column subarray can be collapsed into an OA($N, 2, s^2 \times s, 2$) and an OA($N, 2, s \times s^2, 2$), where collapsing s^2 levels into s levels is done by $\lfloor x/s \rfloor$ for $x \in \mathbb{Z}_{s^2}$, with $\lfloor x \rfloor$ the largest integer not exceeding x [8].

An SOA($N, m, s^2, 2+$) is called column-orthogonal if the inner product between any two columns of the centered array is 0, where centring an SOA($N, m, s^2, 2+$) means that each entry $a \in \mathbb{Z}_{s^2}$ is transformed into $2a - s^2 + 1$ [32]. Denote a column-orthogonal SOA($N, m, s^2, 2+$) by OSOA($N, m, s^2, 2+$).

2.2 Two examples

Before discussing the general construction methods, we first present two constructed designs and their favorable properties as illustrative examples.

Example 2.2. Using the construction method outlined in Section 3, we see that a new SOA(64, 9, 16, 2+), denoted by D, is obtained and presented in Table 1. Compared with the SOA(64, 8, 16, 2+), i.e., D', constructed by He et al. [8], it can accommodate one additional factor. We also compare all the eight-column subarrays of D with the design D'. It is observed that for each eight-column subarray of D, out of its 56 three-dimensional projections, 48 achieve stratifications on $3 \times 3 \times 3$ grids. In contrast, among the 56 three-dimensional projections of D', only 45 can achieve stratifications on $3 \times 3 \times 3$ grids. This indicates that each eight-column subarray of D demonstrates superior three-dimensional projection uniformity compared with D'. For details on the construction of D, please refer to Example 3.5.

Example 2.3. Table 2 displays a novel OSOA(54, 12, 9, 2+), denoted by D, which has not been documented in any literature. This design is constructed using the method explained in Section 4. Compared with the OSOA(54, 7, 9, 2+) from Zhou and Tang [32], it has five more factors. Let D' be the array formed by the first five columns of D. In [10], a strength-three strong orthogonal array of 54 runs, 5 columns and 27 levels was developed, which accomplishes stratifications on 9×3 and 3×9 grids in any two dimensions and $3 \times 3 \times 3$ grids in any three dimensions. Our design D', as a subarray of

D, has the same number of factors and achieves the same stratifications in two or three dimensions. As the maximum number m of factors in an $\mathrm{OA}(54, m, 3, 3)$ is five [11], D' attains the maximum number of factors. For details on the construction of D, please refer to Example 4.11.

Table 1 A new $\mathrm{SOA}(64, 9, 16, 2+)$ (transposed)

0	2	3	1	0	2	3	1	0	2	3	1	0	2	3	1	5	7	6	4	5	7	6	4	5	7	6	4	5	7	6	4
0	8	12	4	7	15	11	3	9	1	5	13	14	6	2	10	1	9	13	5	6	14	10	2	8	0	4	12	15	7	3	11
0	12	4	8	7	11	3	15	9	5	13	1	14	2	10	6	1	13	5	9	6	10	2	14	8	4	12	0	15	3	11	7
0	4	8	12	7	3	15	11	9	13	1	5	14	10	6	2	1	5	9	13	6	2	14	10	8	12	0	4	15	11	7	3
0	2	3	1	5	7	6	4	10	8	9	11	15	13	12	14	5	7	6	4	0	2	3	1	15	13	12	14	10	8	9	11
0	2	3	1	10	8	9	11	15	13	12	14	5	7	6	4	5	7	6	4	15	13	12	14	10	8	9	11	0	2	3	1
0	8	12	4	13	5	1	9	6	14	10	2	11	3	7	15	4	12	8	0	9	1	5	13	2	10	14	6	15	7	3	11
0	12	4	8	13	1	9	5	6	10	2	14	11	7	15	3	4	8	0	12	9	5	13	1	2	14	6	10	15	3	11	7
0	4	8	12	13	9	5	1	6	2	14	10	11	15	3	7	4	0	12	8	9	13	1	5	2	6	10	14	15	11	7	3
10	8	9	11	10	8	9	11	10	8	9	11	10	8	9	11	15	13	12	14	15	13	12	14	15	13	12	14	15	13	12	14
2	10	14	6	5	13	9	1	11	3	7	15	12	4	0	8	3	11	15	7	4	12	8	0	10	2	6	14	13	5	1	9
2	14	6	10	5	9	1	13	11	7	15	3	12	0	8	4	3	15	7	11	4	8	0	12	10	6	14	2	13	1	9	5
2	6	10	14	5	1	13	9	11	15	3	7	12	8	4	0	3	7	11	15	4	0	12	8	10	14	2	6	13	9	5	1
10	8	9	11	15	13	12	14	0	2	3	1	5	7	6	4	15	13	12	14	10	8	9	11	5	7	6	4	0	2	3	1
10	8	9	11	0	2	3	1	5	7	6	4	15	13	12	14	15	13	12	14	5	7	6	4	0	2	3	1	10	8	9	11
8	0	4	12	5	13	9	1	14	6	2	10	3	11	15	7	12	4	0	8	1	9	13	5	10	2	6	14	7	15	11	3
8	4	12	0	5	9	1	13	14	2	10	6	3	15	7	11	12	0	8	4	1	13	5	9	10	6	14	2	7	11	3	15
8	12	0	4	5	1	13	9	14	10	6	2	3	7	11	15	12	8	4	0	1	5	9	13	10	14	2	6	7	3	15	11

Table 2 A new $\mathrm{OSOA}(54, 12, 9, 2+)$ (transposed)

0	4	8	8	0	4	6	1	5	5	6	1	3	7	2	2	3	7	2	3	7	7	2	3	8	0	4
0	8	4	8	4	0	6	5	1	5	1	6	3	2	7	2	7	3	2	7	3	7	3	2	8	4	0
0	0	0	0	0	0	7	7	7	7	7	7	5	5	5	5	5	5	8	8	8	8	8	8	3	3	3
0	4	4	0	8	8	5	6	6	5	1	1	7	2	2	7	3	3	8	0	0	8	4	4	1	5	5
0	8	0	4	8	4	5	1	5	6	1	6	7	3	7	2	3	2	8	4	8	0	4	0	1	6	1
0	4	4	0	8	8	6	1	1	6	5	5	3	7	7	3	2	2	2	3	3	2	7	7	8	0	0
0	8	0	4	8	4	6	5	6	1	5	1	3	2	3	7	2	7	2	7	2	3	7	3	8	4	8
0	4	8	8	0	4	5	6	1	1	5	6	7	2	3	3	7	2	8	0	4	4	8	0	1	5	6
0	8	4	8	4	0	5	1	6	1	6	5	7	3	2	3	2	7	8	4	0	4	0	8	1	6	5
0	0	8	4	4	8	6	6	5	1	1	5	3	3	2	7	7	2	2	2	7	3	3	7	8	8	4
0	0	8	4	4	8	5	5	1	6	6	1	7	7	3	2	2	3	8	8	4	0	0	4	1	1	6
0	6	3	0	6	3	0	6	3	0	6	3	0	6	3	0	6	3	2	8	5	2	8	5	2	8	5
4	8	0	5	6	1	1	5	6	1	5	6	6	1	5	7	2	3	3	7	2	4	8	0	0	4	8
4	0	8	5	1	6	1	6	5	1	6	5	6	5	1	7	3	2	3	2	7	4	0	8	0	8	4
3	3	3	1	1	1	1	1	1	4	4	4	4	4	4	2	2	2	2	2	2	6	6	6	6	6	6
1	6	6	3	7	7	3	2	2	4	8	8	4	0	0	6	1	1	6	5	5	2	3	3	2	7	7
5	6	5	3	2	3	7	2	7	4	0	4	8	0	8	6	5	6	1	5	1	2	7	2	3	7	3
8	4	4	5	6	6	5	1	1	1	5	5	1	6	6	7	2	2	7	3	3	4	8	8	4	0	0
0	4	0	5	1	5	6	1	6	1	6	1	5	6	5	7	3	7	2	3	2	4	0	4	8	0	8
6	1	5	3	7	2	2	3	7	4	8	0	0	4	8	6	1	5	5	6	1	2	3	7	7	2	3
6	5	1	3	2	7	2	7	3	4	0	8	0	8	4	6	5	1	5	1	6	2	7	3	7	3	2
0	0	4	5	5	1	6	6	1	1	1	6	5	5	6	7	7	3	2	2	3	4	4	0	8	8	0
5	5	6	3	3	2	7	7	2	4	4	0	8	8	0	6	6	5	1	1	5	2	2	7	3	3	7
2	8	5	2	8	5	2	8	5	1	7	4	1	7	4	1	7	4	1	7	4	1	7	4	1	7	4

3 Construction of strong orthogonal arrays of strength two plus

In this section, we discuss the construction of an $\mathrm{SOA}(\lambda s^n, m, s^2, 2+)$, where n is an integer greater than two, s is a prime power, and λ is a positive integer that is not divisible by s. Since a construction for $s=2$ has been provided by Cheng et al. [5], we focus on the cases where $s>2$.

3.1 Construction method

To obtain a strong orthogonal array of strength 2+, the following result from He et al. [8] is required.

Lemma 3.1. *An* $\mathrm{SOA}(\lambda s^n, m, s^2, 2+)$ D *exists if and only if there exist two arrays* A *and* B*, where* $A=(a_1,\dots,a_m)$ *is an* $\mathrm{OA}(\lambda s^n, m, s, 2)$ *and* $B=(b_1,\dots,b_m)$ *is an* $\mathrm{OA}(\lambda s^n, m, s, 1)$ *such that* (a_j, a_k, b_k) *is an* $\mathrm{OA}(3)$ *for any* $j \neq k$*. The three arrays are linked through* $D = sA+B$ *if* A *and* B *both have entries from* $\mathbb{Z}_s$*.*

Based on Lemma 3.1, we require an $\mathrm{OA}(\lambda s^n, m, s, 2)$ $A=(a_1,\dots,a_m)$ and an $\mathrm{OA}(\lambda s^n, m, s, 1)$ $B=(b_1,\dots,b_m)$ satisfying the condition that (a_j,a_k,b_k) is an $\mathrm{OA}(3)$ for any $j\neq k$. We select columns from an $\mathrm{OA}(\lambda s^n, m', s, 2)$, i.e., D_1, to construct such A and B, where D_1 has its entries from $\mathrm{GF}(s)$. Then an $\mathrm{SOA}(\lambda s^n, m, s^2, 2+)$ is obtained through

$$D = s\phi(A) + \phi(B), \tag{3.1}$$

where

$$\phi(\alpha_j) = j \tag{3.2}$$

for $j=0,\dots,s-1$ represents a one-to-one mapping from $\mathrm{GF}(s)$ to $\mathbb{Z}_s$, and $\phi(A) = (\phi(a_{ij}))$ for any matrix $A=(a_{ij})$. In this article, D_1 is called the base orthogonal array of the resulting design D.

Suppose that A_0 is a saturated regular $\mathrm{OA}(s^{n-1}, (s^{n-1}-1)/(s-1), s, 2)$ with a generator matrix L. It is worth noting that the first nonzero element in each column of L is always 1. Then we can partition A_0 and L into five parts as follows:

$$A_0 = (A^{(1)},\dots,A^{(5)}) \quad \text{and} \quad L=(L^{(1)},\dots,L^{(5)}), \tag{3.3}$$

where $L^{(k)}$ is the generator matrix of $A^{(k)}$ for $1 \leqslant k \leqslant 5$. Furthermore, $L^{(1)},\dots,L^{(5)}$ consist of vectors $l=(l_1,\dots,l_{n-1})^{\mathrm{T}}$ satisfying the following conditions:

(i) $l_1=0$, and $(l_2,\dots,l_{n-1})$ contains α_{s-2};
(ii) $l_1=0$, and $(l_2,\dots,l_{n-1})$ does not contain α_{s-2};
(iii) $l_1=1$, and $(l_2,\dots,l_{n-1})$ contains α_{s-2} and 1;
(iv) $l_1=1$, and $(l_2,\dots,l_{n-1})$ contains α_{s-2} but not 1;
(v) $l_1=1$, and $(l_2,\dots,l_{n-1})$ does not contain α_{s-2}.

Let $D_0=(d_1,\dots,d_c)$ be a $\mathrm{D}(\lambda s, c, s)$ based on $\mathrm{GF}(s)$, where $d_1 = 0_{\lambda s}$ by assumption and $d_2,\dots,d_c$ are $\mathrm{OA}(\lambda s, 1, s, 1)$ by definition.

Let H be an $\mathrm{OA}(\lambda s, 1, s, 1)$ based on $\mathrm{GF}(s)$ and

$$D_1 = (A_0 \oplus D_0, 0_{s^{n-1}} \oplus H). \tag{3.4}$$

Then D_1 is an $\mathrm{OA}(\lambda s^n, c(s^{n-1}-1)/(s-1)+1, s, 2)$ by Lemma 2.1.

Now we construct A and B from the base orthogonal array D_1 in (3.4) as follows. Let

$$A = [(A^{(1)}, A^{(3)}, A^{(5)}) \oplus d_1, (A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}) \oplus (d_2,\dots,d_c)]. \tag{3.5}$$

Denote by $a_i^{(q)}$ and $l_i^{(q)} = (l_{1,i}^{(q)},\dots,l_{n-1,i}^{(q)})^{\mathrm{T}}$ the i-th columns of $A^{(q)}$ and $L^{(q)}$, respectively. Then B is constructed as follows.

(i) For the column $a_i^{(1)} \oplus d_1 \in A$, let B select $a_{i'}^{(2)} \oplus d_1$ such that $l_{k,i'}^{(2)}$, $k>1$ equals 1 if $l_{k,i}^{(1)} = \alpha_{s-2}$ and 0 otherwise.

(ii) For the column $a_i^{(3)} \oplus d_1 \in A$, let B select $a_{i'}^{(2)} \oplus d_1$ such that $l_{k,i'}^{(2)}$, $k > 1$ equals 1 if $l_{k,i}^{(1)} = 1$ and 0 otherwise.

(iii) For the column $a_i^{(5)} \oplus d_1 \in A$, let B select $a_i^{(5)} \oplus d_2$.

(iv) For the column $a_i^{(q)} \oplus d_j \in A$ with $q \in \{1, 2\}$ and $j > 1$, let B select $a_{i'}^{(4)} \oplus d_1$ such that $l_{k,i'}^{(4)}$, $k > 1$ equals 0 if $l_{k,i}^{(q)} = 0$ and α_{s-2} otherwise.

(v) For the column $a_i^{(q)} \oplus d_j \in A$ with $q \in \{3, 4\}$ and $j > 1$, let B select $a_{i'}^{(2)} \oplus d_1$ such that $l_{k,i'}^{(2)}$ $k > 1$ equals 1 if $l_{k,i}^{(q)} = \alpha_{s-2}$ and 0 otherwise.

Theorem 3.2. *Let A and B be constructed as described above. Then the design $D = s\phi(A) + \phi(B)$ is an $\mathrm{SOA}(\lambda s^n, m, s^2, 2+)$, where*

$$m = c(s^{n-1} - 1)/(s - 1) - c(s - 1)^{n-2} - ((s - 1)^{n-2} - 1)/(s - 2) + (s - 2)^{n-2}.$$

The construction method divides L and A_0 into five segments as illustrated in (3.3). With the exception of $L^{(1)}$ and $L^{(3)}$, the remaining portions, i.e., $L^{(2)}$, $L^{(4)}$ and $L^{(5)}$, become nonempty when $n = 3$, and all five segments are nonempty for $n > 3$. Leveraging Lemma 3.1, it is feasible to generate a strong orthogonal array of strength 2+ with a reduced number of columns by utilizing any subset of these segments. The resulting design will form a subarray of the one generated by Theorem 3.2.

Combining Theorem 3.2 with the existence results for $\mathrm{D}(\lambda s, \lambda s, s)$ in [12, Chapter 6] yields the following corollary, whose proof is provided in Appendix A.

Corollary 3.3. *There exists an $\mathrm{SOA}(\lambda s^n, m, s^2, 2+)$ with*

$$m = \lambda(s^n - 1)/(s - 1) - (\lambda(s - 1)^n - (\lambda - 1)(s - 1)^{n-2} - 1)/(s - 2) + (s - 2)^{n-2} - \lambda,$$

if one of the following conditions holds:

(i) $s = p^{u_1}$, $\lambda = kp^{u_2}$, $k \in \{1, 2, 4\}$, *p is a prime, $p \nmid k$, and $u_1 > u_2 \geqslant 0$ are integers*;

(ii) $s - 1$ *is a prime power and* $\lambda = (s - 1)^k$ *with* $k \geqslant 1$.

Remark 3.4. For the case of $\lambda = 1$, He et al. [8, Theorem 4] proposed a construction method using regular orthogonal arrays. As a comparison, our method gives $(s - 2)^{n-2} - 1$ more factors.

Example 3.5 (Example 2.2 continued)**.** The $\mathrm{SOA}(64, 9, 16, 2+)$ D in Table 1 is constructed by the following procedures. Let $\lambda = 1$, $s = 4$ and $n = 3$. Take $\mathrm{GF}(4) = \{\alpha_0 = 0, \alpha_1 = x, \alpha_2 = 1 + x, \alpha_3 = 1\}$ with a primitive polynomial $p(x) = x^2 + x + 1$. Let $A_0 = (a_1, a_2, \ldots, a_5)$ be a Rao-Hamming $\mathrm{OA}(16, 5, 4, 2)$ with the generator matrix

$$L = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & x+1 & 0 & 1 & x \end{pmatrix}. \tag{3.6}$$

Partition L as

$$(L^{(2)}, L^{(4)}, L^{(5)}) = \left(\begin{array}{c|c|ccc} 0 & 1 & 1 & 1 & 1 \\ 1 & x+1 & 0 & 1 & x \end{array}\right). \tag{3.7}$$

When $l = (l_1, l_2)^{\mathrm{T}} \in [\mathrm{GF}(4)]^2$ has its first nonzero element being 1, l cannot satisfy (i) $l_1 = 0$ and $l_2 = \alpha_{s-2}$ or (ii) $l_1 = 1$, and l_2 contains both α_{s-2} and 1. This means $L^{(1)} = L^{(3)} = \emptyset$. According to the partition of L, we divide A_0 into

$$A_0 = (A^{(2)}, A^{(4)}, A^{(5)}), \tag{3.8}$$

where $A^{(2)} = a_1$, $A^{(4)} = a_2$ and $A^{(5)} = (a_3, a_4, a_5)$. Let $D_0 = (d_1, d_2, d_3, d_4)$ be the $\mathrm{D}(4, 4, 4)$ in Table B.1 in Appendix B. With columns selected from $A_0 \oplus D_0$, our construction gives two arrays

$$A = [(a_3, a_4, a_5) \oplus d_1, (a_1, a_2) \oplus (d_2, d_3, d_4)]$$

and

$$B = [(a_3, a_4, a_5) \oplus d_2, (a_2, a_1) \oplus (d_1, d_1, d_1)].$$

Then $D = 4\phi(A) + \phi(B)$ is the $\mathrm{SOA}(64, 9, 16, 2+)$ shown in Table 1.

Table 3 A comparison of the number m of factors for the constructed SOA$(N, m, s^2, 2+)$ and the number m' of factors for the existing SOA$(N, m', s^2, 2+)$, where $N = \lambda s^n$

N	λ	s	n	m	m'	Source
27	1	3	3	6	6	He et al. [8]
81	1	3	4	25	25	He et al. [8]
243	1	3	5	90	90	He et al. [8]
64	1	4	3	9	8	He et al. [8]
256	1	4	4	48	45	He et al. [8]
1,024	1	4	5	227	220	He et al. [8]
125	1	5	3	12	10	He et al. [8]
625	1	5	4	79	71	He et al. [8]
3,125	1	5	5	466	440	He et al. [8]
54	2	3	3	12	12	Chen and Tang [4]
162	2	3	4	52	43	Jiang et al. [13]
486	2	3	5	186	165	Jiang et al. [13]
128	2	4	3	17	–	–
512	2	4	4	96	–	–
2,048	2	4	5	459	–	–
250	2	5	3	22	15	Jiang et al. [13]
1,250	2	5	4	154	118	Jiang et al. [13]
6,250	2	5	5	926	759	Jiang et al. [13]
192	3	4	3	25	–	–
768	3	4	4	144	–	–
3,072	3	4	5	691	–	–
108	4	3	3	24	–	–
324	4	3	4	106	–	–
972	4	3	5	378	–	–

Example 3.6. Consider the scenario where $\lambda = 3$, $s = 4$ and $n = 3$, implying $N = \lambda s^n = 192$. Let A_0 be the regular OA$(16, 5, 4, 2)$ presented in Example 3.5, and D_0 be D$(12, 12, 4)$ provided in Table B.1. By applying Theorem 3.2, we can construct an SOA$(192, 25, 16, 2+)$, denoted by D, that is derived from A_0 and D_0. This design attains stratifications on 16×4 and 4×16 grids in two dimensions. According to [8, Theorem 5], another strong orthogonal array of strength 2+, denoted by D', with 192 runs, 21 factors and 12 levels can be constructed using the OA$(64, 21, 4, 2)$ available on the N.J.A. Sloane website (http://neilsloane.com/oadir/). Compared with D, D' has 4 fewer factors and achieves stratifications on 12×4 and 4×12 grids in two dimensions.

We also compare the $12,650$ 21-column subarrays of D with the design D' using the uniform projection criterion, which was proposed by Sun et al. [28] to evaluate the two-dimensional projection uniformity of designs. The criterion values, where smaller values are better, for the 21-column subarrays of D are distributed in the interval $(0.000728, 0.000734)$, while the criterion value for D' is 0.0013. This indicates that all 21-column subarrays of D outperform D' under the uniform projection criterion.

Table 3 compares the number of factors for the SOA$(\lambda s^n, m, s^2, 2+)$ constructed using our method and the existing SOA$(\lambda s^n, m', s^2, 2+)$. Here, the symbol '–' indicates a lack of existing designs or methods. In contrast to the approach proposed by [8], which used regular orthogonal arrays, our construction employs designs in (3.4) as base orthogonal arrays, rendering it more versatile and general. Notably, when $\lambda = 1$, our method generates $(s-2)^{n-2}-1$ more columns compared with the method introduced by He et al. [8]. Furthermore, when $\lambda > 1$, our method remains applicable, whereas that of He et al. [8] is not. In the case of $\lambda = 2$ and odd s, our method outperforms the strategy outlined in [13]. For $n = 3$ and $\lambda = 2$, Chen and Tang [4] obtained an SOA$(54, 12, 9, 2+)$ with identical parameters to ours using a computer-based search. Notably, our method can generate new strong orthogonal arrays in all the other scenarios where existing methods are inapplicable. Interested readers can access all the designs obtained through our method in

Table 3 at https://github.com/bcjiang0326/data. The difference schemes in Table B.1 in Appendix B are employed for constructing these designs.

3.2 Completeness of strong orthogonal arrays of strength two plus

In this subsection, we explore the possibility of further increasing the number of columns for the constructed strong orthogonal arrays of strength 2+ in Subsection 3.1. Assume that we have an $\mathrm{SOA}(\lambda s^n, m, s^2, 2+)$, denoted by D, constructed from A and B through the equation $D = s\phi(A) + \phi(B)$, as outlined in Theorem 3.2. In this construction, the columns of both A and B are selected from the base orthogonal array D_1 in (3.4).

Given that the construction of A plays a pivotal role, we investigate the possibility of adding more columns from D_1 to A while ensuring the existence of a matrix B' such that (a_i, a_j, b'_i) forms an OA(3) for any $i \neq j$, where b'_i is the i-th column of B' and is still selected from D_1.

Definition 3.7 (Completeness)**.** If no column of D_1 can be added to A to construct strong orthogonal arrays of strength 2+, the design $D = s\phi(A) + \phi(B)$ is said to be *complete* within D_1.

A property of difference schemes with entries from $\mathrm{GF}(s)$ is needed. For simplicity and convenience, we name this property as the property δ.

Definition 3.8 (Property δ of difference schemes)**.** For any $\mathrm{D}(\lambda s, c, s)$ $D_0 = (d_1, \ldots, d_c)$ with entries from $\mathrm{GF}(s)$, we say that D_0 has the *property* δ if for any $d_j \in D_0$ and any $\beta \in \mathrm{GF}(s)$, there exists a column $d_{j'} \in D_0$ (which can be the same as d_j) such that $d_{j'} - \beta d_j$ is not an $\mathrm{OA}(\lambda s, 1, s, 1)$.

The property δ for D_0 is quite mild. In fact, D_0 must have the property δ if it cannot be expanded into a $\mathrm{D}(\lambda s, c+1, s)$. To illustrate this, assume that there exist a column $d_j \in D_0$ and an element $\beta \in \mathrm{GF}(s)$ such that for all $d_{j'} \in D_0$, $d_{j'} - \beta d_j$ is an $\mathrm{OA}(\lambda s, 1, s, 1)$. This implies that $\beta d_j \notin D_0$, and we can add βd_j to D_0 to obtain a $\mathrm{D}(\lambda s, c+1, s)$. It is important to note that $c_1 = \lambda s$ represents the maximum value of c for which a $\mathrm{D}(\lambda s, c, s)$ can exist. The following conclusion is straightforward.

Proposition 3.9. *For any* $\mathrm{D}(\lambda s, c, s)$ *with entries from* $\mathrm{GF}(s)$*, it either possesses the property* δ *or can be extended to have the property* δ*.*

We have the following conclusion, whose proof is provided in Appendix A.

Theorem 3.10. *The strong orthogonal array of strength* 2+ *obtained in Theorem* 3.2 *is complete if the used* D_0 *possesses the property* δ*.*

Remark 3.11. Based on Proposition 3.9 and Theorem 3.10, we can directly conclude that if we select a $\mathrm{D}(\lambda s, \lambda s, s)$ for D_0, the strong orthogonal array constructed in Theorem 3.2 is complete. So the strong orthogonal arrays in Corollary 3.3 are all complete.

Remark 3.12. When D_0 is a $\mathrm{D}(\lambda s, \lambda s, s)$, completeness guarantees that the design constructed in Subsection 3.1 cannot accommodate any additional columns, a desirable attribute. However, it is important to recognize that completeness does not confirm that our construction approach achieves the maximum possible number of columns. There might exist an alternative construction method capable of producing a design with a greater number of columns.

4 Construction of column-orthogonal strong orthogonal arrays of strength two plus

In this section, our goal is to construct an $\mathrm{OSOA}(\lambda s^n, m, s, 2+)$, where n is an integer greater than two, s is a prime power, and λ is a positive integer that is not divisible by s. Given that Zhou and Tang [32] have already provided the construction method in the case of $s = 2$, achieving the maximum number of columns in the resulting design, our emphasis is on scenarios where $s > 2$. The following lemma from Zhou and Tang [32] is a key result for constructing an $\mathrm{OSOA}(\lambda s^n, m, s, 2+)$.

Lemma 4.1. *Suppose that there exist two* $\mathrm{OA}(\lambda s^n, m, s, 2)$ $A = (a_1, \ldots, a_m)$ *and* $B = (b_1, \ldots, b_m)$ *such that* (a_j, a_k, b_k) *is an* $\mathrm{OA}(3)$ *for any* $j \neq k$. *If both* A *and* B *have entries from* $\mathbb{Z}_s$, $D = sA + B$ *is an* $\mathrm{OSOA}(\lambda s^n, m, s^2, 2+)$.

In contrast to Lemma 3.1, B in Lemma 4.1 has strength 2, and thus disallowing repeated columns. Similar to Subsection 3.1, we construct A and B through a base orthogonal array $\mathrm{OA}(\lambda s^n, m', s, 2)$ over $\mathrm{GF}(s)$, and then the desired $\mathrm{OSOA}(\lambda s^n, m, s^2, 2+)$ is obtained using (3.1).

Let $A_0 = (a_1, \ldots, a_{s+1})$ be a regular $\mathrm{OA}(s^2, s+1, s, 2)$ with a generator matrix $L = (l_1, \ldots, l_{s+1})$. Without loss of generality, assume that the first four columns of L are given by

$$(l_1, l_2, l_3, l_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & \alpha_{s-2} & 1 & 1+\alpha_{s-2} \end{pmatrix}. \tag{4.1}$$

Let $D^{(1)} = (d_1^{(1)}, \ldots, d_c^{(1)})$ be a $\mathrm{D}(\lambda s, c, s)$ with entries from $\mathrm{GF}(s)$ and $d^{(1)} = 0_{\lambda s}$. Define $v = (0, 1, \alpha_1, \ldots, \alpha_{s-2})^{\mathrm{T}}$, $V = v \cdot v^{\mathrm{T}}$ and

$$D^{(k)} = V \oplus D^{(k-1)} \tag{4.2}$$

for any $k \geqslant 2$. Then V is a $\mathrm{D}(s, s, s)$ and

$$D^{(k)} = (d_1^{(k)}, d_2^{(k)}, \ldots, d_{cs^{k-1}}^{(k)}) \tag{4.3}$$

is a $\mathrm{D}(\lambda s^k, cs^{k-1}, s)$ for any $k \geqslant 1$.

Pick a fixed integer q where $1 \leqslant q \leqslant \lfloor (n-1)/2 \rfloor$. Define

$$H_0 = \begin{cases} 0_\lambda \oplus v, & \text{if } n - 2q = 1, \\ (v \oplus D^{(1)}, 0_{\lambda s} \oplus v), & \text{if } n - 2q = 2, \\ (\tilde{A} \oplus D^{(1)}, 0_{\lambda s^{n-2q-1}} \oplus v), & \text{if } n - 2q \geqslant 3, \end{cases} \tag{4.4}$$

where $\tilde{A}$ is a regular $\mathrm{OA}(s^{n-2q-1}, (s^{n-2q-1}-1)/(s-1), s, 2)$ when $n-2q \geqslant 3$. Then H_0 is an $\mathrm{OA}(\lambda s, 1, s, 1)$ if $n - 2q = 1$, and an $\mathrm{OA}(\lambda s^{n-2q}, c(s^{n-2q-1} - 1)/(s-1) + 1, s, 2)$ otherwise.

Recursively define

$$H_k = (A_0 \oplus D^{(n-2q+2k-2)}, 0_{s^2} \oplus H_{k-1}) \tag{4.5}$$

for $1 \leqslant k \leqslant q$. Lemma 2.1 shows that H_k is an

$$\mathrm{OA}(\lambda s^{n-2q+2k}, c(s^{n-2q+2k-1} - 1)/(s-1) + 1, s, 2)$$

for $1 \leqslant k \leqslant q$. In particular, H_q can be expressed as

$$H_q = (A_0 \oplus D^{(n-2)}, 0_{s^2} \oplus A_0 \oplus D^{(n-4)}, \ldots, 0_{s^{2q-2}} \oplus A_0 \oplus D^{(n-2q)}, 0_{s^{2q}} \oplus H_0). \tag{4.6}$$

Now we construct A and B through the base orthogonal array H_q in (4.6) as follows.

Step 1. For all $1 \leqslant k \leqslant q$, let A and B be constructed by stacking the matrices

$$0_{s^{2k-2}} \oplus [(a_1, a_2) \oplus (d_2^{(n-2k)}, \ldots, d_{cs^{n-2k-1}}^{(n-2k)}), a_3 \oplus d_1^{(n-2k)}] \tag{4.7}$$

and

$$0_{s^{2k-2}} \oplus [(a_4, a_3) \oplus (d_2^{(n-2k)}, \ldots, d_{cs^{n-2k-1}}^{(n-2k)}), a_2 \oplus d_1^{(n-2k)}], \tag{4.8}$$

respectively, by columns.

Step 2. Randomly select r_q distinct columns from the matrix $0_{s^{2q}} \oplus H_0$ and append them to A by columns, where

$$r_q = \min\{c(s^{n-2q-1} - 1)/(s-1) + 1, c(s-3)(s^{n-1} - s^{n-2q-1})/(s^2 - 1) + q\}. \tag{4.9}$$

Subsequently, randomly select r_q distinct columns from the matrices

$$0_{s^{2k-2}} \oplus [a_4 \oplus d_1^{(n-2k)}, (a_5, \ldots, a_{s+1}) \oplus D^{(n-2k)}], \quad 1 \leqslant k \leqslant q, \tag{4.10}$$

and append them to B by columns.

Theorem 4.2. *For any integer* $1 \leqslant q \leqslant \lfloor (n-1)/2 \rfloor$*, let* A *and* B *be constructed as described above. Then the design* $D = s\phi(A) + \phi(B)$ *is an* $\mathrm{OSOA}(\lambda s^n, m_q, s^2, 2+)$ *with*

$$m_q = 2c(s^{n-1} - s^{n-2q-1})/(s^2-1) - q + r_q, \tag{4.11}$$

where $\phi(\cdot)$ *and* r_q *are defined in* (3.2) *and* (4.9)*, respectively.*

Define $q_0 = \lfloor (n-1)/2 \rfloor$. By maximizing m_q in (4.11) for $1 \leqslant q \leqslant q_0$, we can get the optimal q^* and the corresponding m_{q^*} values.

Corollary 4.3. (i) *If* $s = 3$ *and* $n \leqslant 4(c+1)$*,*

$$q^* = q_0 \quad \textit{and} \quad m_{q^*} = m_{q_0} = c(3^n - 3^{n-2q_0})/12 - q_0 + r_{q_0},$$

where $r_{q_0} = \min\{c+1, q_0\}$ *if* n *is even and* $r_{q_0} = 1$ *otherwise.*

(ii) *If* $s > 3$*,*

$$q^* = 1 \quad \textit{and} \quad m_{q^*} = m_1 = 2cs^{n-3} + c(s^{n-3}-1)/(s-1).$$

Remark 4.4. Note that the optimal value of q^* depends on the parameters s, n and c. If $s = 3$ and $n \leqslant 4(c+1)$, $q^* = q_0$, which is the largest possible value of q. On the other hand, if $s > 3$, $q^* = 1$, which is the smallest possible value of q.

Remark 4.5. The existence of $\mathrm{D}(3,3,3)$ guarantees the existence of $\mathrm{D}(3\lambda,3,3)$. Therefore, for $s = 3$, we can always assume that $c \geqslant 3$. Corollary 4.3(i) then implies that $q^* = q_0$ when $s = 3$ and $n \leqslant 16$.

Similar to Theorem 3.2, when a $\mathrm{D}(\lambda s, \lambda s, s)$ exists, Theorem 4.2 has the following corollary.

Corollary 4.6. *There exists an* $\mathrm{OSOA}(\lambda s^n, m, s^2, 2+)$ *with*

$$m = \begin{cases} \lambda(3^n - 3^{n-2q_0})/4 - q_0 + r_{q_0}, & \textit{if } s = 3, \\ 2\lambda s^{n-2} + \lambda(s^{n-2}-1)/(s-1) - \lambda, & \textit{if } s > 3, \end{cases}$$

where $q_0 = \lfloor (n-1)/2 \rfloor$ *and* $r_{q_0} = \min\{3\lambda+1, q_0\}$ *if* n *is even and* $r_{q_0} = 1$ *otherwise, if one of the following conditions holds*:

(i) $s = p^{u_1}$, $\lambda = kp^{u_2}$, $k \in \{1,2,4\}$, *p is a prime,* $p \nmid k$ *and* $u_1 > u_2 \geqslant 0$ *are integers*;

(ii) $s-1$ *is a prime power and* $\lambda = (s-1)^k$ *with* $k \geqslant 1$.

Remark 4.7. If a $\mathrm{D}(\lambda s, \lambda s, s)$ exists, [32, Theorem 4] can be used to construct an

$$\mathrm{OSOA}(\lambda s^n, m', s^2, 2+)$$

with $m' = \lambda(s^{n-1}-1)/(s-1) - \lambda + 1$. In comparison, when $s > 3$, Corollary 4.6 provides $\lambda s^{n-2} - 1$ additional factors. For $s = 3$, it provides at least $\lambda(3^{n-1} - 3^{n-2q_0} + 6)/4 + r_{q_0} - q_0 - 1$ additional factors.

Corollary 4.8. *There is an* $\mathrm{OSOA}(27\lambda, 6\lambda, 9, 2+)$ *for any integers* $\lambda = 2^k$ *and* $k \geqslant 0$.

Remark 4.9. For $n = s = 3$ and $\lambda \in \{1,2\}$, if we juxtapose the constructed matrices A and B in Theorem 4.2, we obtain an $\mathrm{OA}(27\lambda, 12\lambda, 3, 2)$, which is only one column short of being maximal. This implies that no columns can be added to A and B while maintaining the requirements in Lemma 4.1. Therefore, among all $\mathrm{OSOA}(27\lambda, m, 9, 2+)$ where $\lambda \in \{1,2\}$ derived from A and B as in Lemma 4.1, our approach achieves the maximum number of columns.

Example 4.10. Let $\lambda = 1$, $s = 3$ and $n = 3$. Then we have $q = q_0 = \lfloor (n-1)/2 \rfloor = 1$ and $r_q = 1$ by (4.9). Let

$$\mathrm{GF}(3) = \{\alpha_0 = 0, \alpha_1 = 2, \alpha_2 = 1\}.$$

Note that

$$1 + \alpha_{s-2} = 1 + \alpha_1 = 0.$$

Let A_0 be the regular $\mathrm{OA}(9,4,3,2)$ with the generator matrix

$$L = (l_1, l_2, l_3, l_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}. \tag{4.12}$$

Let $H_0 = (\alpha_0, \alpha_1, \alpha_2)^{\mathrm{T}}$ and $D^{(1)} = H_0 \cdot H_0^{\mathrm{T}} = (d_1^{(1)}, d_2^{(1)}, d_3^{(1)})$ be a $\mathrm{D}(3,3,3)$. Then

$$H_1 = (A_0 \oplus D^{(1)}, 0_9 \oplus H_0)$$

is an $\mathrm{OA}(27, 13, 3, 2)$. Based on H_1, our construction gives two arrays

$$A = [(a_1, a_2) \oplus (d_2^{(1)}, d_3^{(1)}), a_3 \oplus d_1^{(1)}, 0_9 \oplus H_0]$$

and

$$B = [(a_4, a_3) \oplus (d_2^{(1)}, d_3^{(1)}), a_2 \oplus d_1^{(1)}, a_4 \oplus d_1^{(1)}].$$

The first five columns of A and B are from Step 1, and the last ones are from Step 2. By Theorem 4.2, the array $D = 3\phi(A) + \phi(B)$, shown in Table 4, is an $\mathrm{OSOA}(27, 6, 9, 2+)$ and has not been reported in the literature.

Example 4.11 (Example 2.3 continued)**.** The $\mathrm{OSOA}(54, 12, 9, 2+)$ D in Table 2 is constructed by the following procedures. Let $\lambda = 2$, $s = 3$ and $n = 3$. Then $q = q_0 = \lfloor (n-1)/2 \rfloor = 1$ and $r_q = 1$ by (4.9). Take

$$\mathrm{GF}(3) = \{\alpha_0 = 0, \alpha_1 = 2, \alpha_2 = 1\}.$$

Let $A_0 = (a_1, a_2, a_3, a_4)$ be the regular $\mathrm{OA}(9, 4, 3, 2)$ used in Example 4.10, whose generator matrix L is given in (4.12). Let $D^{(1)} = (d_1^{(1)}, \ldots, d_6^{(1)})$ be the $\mathrm{D}(6, 6, 3)$ given in Table B.1 in Appendix B and

$$H_0 = (\alpha_0, \alpha_1, \alpha_2, \alpha_0, \alpha_1, \alpha_2)^{\mathrm{T}}.$$

Then

$$H_1 = (A_0 \oplus D^{(1)}, 0_9 \oplus H_0)$$

is an $\mathrm{OA}(54, 25, 3, 2)$. Using H_1, we see that our construction gives two arrays

$$A = [(a_1, a_2) \oplus (d_2^{(1)}, \ldots, d_6^{(1)}), a_3 \oplus d_1^{(1)}, 0_9 \oplus H_0]$$

and

$$B = [(a_4, a_3) \oplus (d_2^{(1)}, \ldots, d_6^{(1)}), a_2 \oplus d_1^{(1)}, a_4 \oplus d_1^{(1)}].$$

Except for the last ones, all the columns of A and B are from Step 1. By Theorem 4.2,

$$D = 3\phi(A) + \phi(B)$$

is an $\mathrm{OSOA}(54, 12, 9, 2+)$, as tabulated in Table 2.

Table 5 compares the number of factors for designs obtained from our method and Zhou and Tang [32]'s method. It can be observed that our method consistently produces more factors than Zhou and Tang [32]'s. All the designs obtained by our method in Table 5 can be accessed at https://github.com/bcjiang 0326/data, for interested readers. The difference schemes in Table B.1 in Appendix B are used for constructing these designs.

Table 4 The transposed $\mathrm{OSOA}(27, 6, 9, 2+)$ constructed in Example 4.10

0	3	6	0	3	6	0	3	6	1	4	7	1	4	7	1	4	7	2	5	8	2	5	8	2	5	8
0	8	4	3	2	7	6	5	1	1	6	5	4	0	8	7	3	2	2	7	3	5	1	6	8	4	0
0	4	8	3	7	2	6	1	5	1	5	6	4	8	0	7	2	3	2	3	7	5	6	1	8	0	4
0	0	0	5	5	5	7	7	7	4	4	4	6	6	6	2	2	2	8	8	8	1	1	1	3	3	3
0	8	4	7	3	2	5	1	6	4	0	8	2	7	3	6	5	1	8	4	0	3	2	7	1	6	5
0	4	8	7	2	3	5	6	1	4	8	0	2	3	7	6	1	5	8	0	4	3	7	2	1	5	6

Table 5 A comparison of the number m of factors for the constructed OSOA$(N, m, s^2, 2+)$ and the number m' of factors for OSOA$(N, m', s^2, 2+)$ in [32], where $N = \lambda s^n$

N	λ	s	n	q	m	m'
27	1	3	3	1	6	4
81	1	3	4	1	18	13
243	1	3	5	2	59	40
64	1	4	3	1	8	5
256	1	4	4	1	36	21
1,024	1	4	5	1	148	85
125	1	5	3	1	10	6
625	1	5	4	1	55	31
3,125	1	5	5	1	280	156
54	2	3	3	1	12	7
162	2	3	4	1	36	25
486	2	3	5	2	119	79
128	2	4	3	1	16	9
512	2	4	4	1	72	41
2,048	2	4	5	1	296	169
250	2	5	3	1	20	11
1,250	2	5	4	1	110	61
6250	2	5	5	1	560	311
192	3	4	3	1	24	13
768	3	4	4	1	108	61
3,072	3	4	5	1	444	253
108	4	3	3	1	24	13
324	4	3	4	1	72	49
972	4	3	5	2	239	157

5 Three-dimensional projections of strong orthogonal arrays

In this section, we investigate the three-dimensional projection properties of the strong orthogonal arrays and column-orthogonal strong orthogonal arrays of strength 2+ constructed in Sections 3 and 4, respectively. For these strength-2+ designs with identical parameters, we can compare and filter them based on their three-dimensional projection properties.

For an SOA$(\lambda s^n, m, s^2, 2+)$ D with $m \geqslant 3$, we define $\pi(D)$ as the proportion of its three-dimensional projections that can be collapsed into an OA$(\lambda s^n, 3, s, 3)$. The larger the value of $\pi(D)$, the more space-filling the design becomes in three-dimensional projections. Specifically, $\pi(D) = 1$ if and only if D can be collapsed into an OA$(\lambda s^n, m, s, 3)$. We observe that for the constructed design D in Theorem 3.2, the value of $\pi(D)$ depends on the used difference scheme $D_0 = (d_1, \ldots, d_c)$. In particular, for $n = 3$, we have the following result.

Theorem 5.1. *For the constructed* SOA$(\lambda s^n, m, s^2, 2+)$ D *in Theorem* 3.2 *with* $n = 3$, *the value of* $\pi(D)$ *is given by*

$$\pi(D) = \frac{6(c-1)(s(s-2) + (s+c-1)(c-2)) + 12\omega_0}{(2c+s-3)(2c+s-4)(2c+s-5)}, \tag{5.1}$$

where ω_0 *is the number of tuples* $(\beta, d_j, d_{j'})$, *with* $\beta \in \mathrm{GF}(s) \setminus \{0, 1\}$ *and* $2 \leqslant j < j' \leqslant c$, *such that* $d_{j'} - \beta d_j$ *is an* OA$(\lambda s, 1, s, 1)$.

Similar to Theorem 5.1, we have the following result for the column-orthogonal strong orthogonal array of strength 2+ constructed in Theorem 4.2.

Theorem 5.2. *For the constructed* OSOA$(\lambda s^n, m, s^2, 2+)$ D *in Theorem* 4.2 *with* $n = 3$, *the value of*

$\pi(D)$ is given by

$$\pi(D) = \frac{3(c-1)(c^2-c+1)+3\omega_1}{2c(c-1)(2c-1)}, \tag{5.2}$$

where ω_1 is the number of pairs $(d_j^{(1)}, d_{j'}^{(1)})$ with $2 \leqslant j, j' \leqslant c$, such that $d_{j'}^{(1)} - (\alpha_{s-2}-1)d_j^{(1)}$ is an $\mathrm{OA}(\lambda s, 1, s, 1)$*. For $s = 3$ in particular, we have*

$$\omega_1 = (c-1)(c-2) \quad \textit{and} \quad \pi(D) = (3c^2-3)/(4c^2-2c).$$

In accordance with Theorems 5.1 and 5.2, we can achieve a strong orthogonal array or column-orthogonal strong orthogonal array of strength 2+ that attains the maximum $\pi(D)$ value by selecting a difference scheme that maximizes ω_0 or ω_1.

Example 5.3. Suppose that we wish to construct an $\mathrm{OSOA}(192, 12, 16, 2+)$ and hope that this design exhibits the best possible three-dimensional space-filling properties, specifically the highest $\pi(D)$ value.

One solution is to choose the $\mathrm{D}(12, 12, 4)$ in Table B.1 as $D^{(1)}$. By applying Theorem 4.2, we can obtain an $\mathrm{OSOA}(192, 24, 16, 2+)$ and then compare the $\pi(D)$ values of all its 12-column subarrays to select the best one. This approach would require calculating the $\pi(D)$ values for a staggering 2,704,156 instances of $\mathrm{OSOA}(192, 12, 16, 2+)$, which represents an exceptionally demanding computational task.

Alternatively, we can select a $\mathrm{D}(12, 6, 4)$ for $D^{(1)}$ and directly obtain an $\mathrm{OSOA}(192, 12, 16, 2+)$ using Theorem 4.2. According to Theorem 5.2, higher values of ω_1 correspond to greater $\pi(D)$ values. Therefore, we only need to compare the ω_1 values among the 462 $\mathrm{D}(12, 6, 4)$ subarrays within the $\mathrm{D}(12, 12, 4)$, where the first column is 0_{12}, resulting in a significantly reduced computational workload. A straightforward calculation reveals that the optimal $\mathrm{D}(12, 6, 4)$ corresponds to a ω_1 value of 15, resulting in a $\pi(D)$ value of 0.77 for the constructed $\mathrm{OSOA}(192, 12, 16, 2+)$.

6 Discussion

We introduce two innovative methods for constructing strong orthogonal arrays and column-orthogonal strong orthogonal arrays of strength 2+. Compared with the existing methods, our approaches can accommodate a greater number of factors with identical run sizes or, at the very least, an equal number. Additionally, our methods allow us to construct a wider variety of new strong orthogonal arrays with run sizes that were previously beyond reach. These designs, obtained through our methods, achieve stratifications on $s^2 \times s$ and $s \times s^2$ grids in any two dimensions, as well as $s \times s \times s$ grids in a substantial portion of three dimensions.

A natural question arises: can the number of columns in the strong orthogonal arrays obtained through our construction methods reach the maximum possible value? To answer this question, we need to derive tight upper bounds on the maximum number of factors in a strong orthogonal array or a column-orthogonal strong orthogonal array of strength 2+. Currently, these remain open problems, even for specific cases, and require exhaustive enumeration of all the potential strong orthogonal arrays or column-orthogonal strong orthogonal arrays.

We also propose several avenues for future research. First, a promising direction is to leverage our base orthogonal arrays to generate other types of space-filling designs [9, 10, 19, 20, 31]. Secondly, while Theorems 5.1 and 5.2 unveil relationships between difference schemes and the three-dimensional projection properties of the obtained strong orthogonal arrays and column-orthogonal strong orthogonal arrays for $n = 3$, future research aims to extend these relationships to cases with $n > 3$ and to identify optimal difference schemes when they are not unique for given parameters. Thirdly, in Examples 2.3 and 4.11, we observe that the $\mathrm{OSOA}(54, 12, 9, 2+)$ obtained using our method can be partitioned into three subarrays with 5, 4 and 3 columns, respectively, each achieving stratifications on $3 \times 3 \times 3$ grids in all three dimensions. However, exhaustive searches confirm that it is impossible to have two subarrays of 5 columns with this property. A potential direction for future research is to group columns in the constructed strong orthogonal arrays into subarrays with enhanced properties, with the concept of fractional factorials of variable resolution [14] being a potentially useful approach.

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Appendix A Proofs

Assume in Lemma 2.1 that $A_0 = (a_1, \dots, a_r)$, where $r = (s^n - 1)/(s - 1)$ is a Rao-Hamming OA$(s^n, r, s, 2)$ with a generator matrix $L = (l_1, \dots, l_r)$, $D_0 = (d_1, \dots, d_c)$ is a D$(\lambda s, c, s)$ and $H = (h_1, \dots, h_k)$ is an OA$(\lambda s, k, s, t)$ with $k = t = 1$ or $k \geqslant t \geqslant 2$, all based on GF$(s)$. Then we obtain the following result.

Lemma A.1. (i) $[a_1 \oplus d_1, 0_{s^n} \oplus (h_1, h_2)]$ *forms an* OA(3).

(ii) $[a_i \oplus d_j, a_{i'} \oplus d_{j'}, 0_{s^n} \oplus h_1]$ *forms an* OA(3) *if* $i \neq i'$, *and the converse is true if* $s \nmid \lambda$.

(iii) $[a_i \oplus d_j, a_{i'} \oplus (d_1, d_2)]$ *forms an* OA(3) *if* $i \neq i'$, *and the converse is true if* $s \nmid \lambda$.

(iv) *If* $\beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 = 0$ *for some nonzero* $\beta_1, \beta_2, \beta_3 \in \mathrm{GF}(s)$, $[a_1 \oplus d_{j_1}, a_2 \oplus d_{j_2}, a_3 \oplus d_{j_3}]$ *forms an* OA(3) *if and only if* $\beta_1 d_{j_1} + \beta_2 d_{j_2} + \beta_3 d_{j_3}$ *forms an* OA$(\lambda s, 1, s, 1)$.

(v) $[a_1 \oplus d_{j_1}, a_2 \oplus d_{j_2}, a_3 \oplus d_{j_3}]$ *forms an* OA(3) *if* l_1, l_2 *and* l_3 *are linearly independent.*

Proof. Let $G = \mathrm{GF}(s)$ and $F = \{1, \dots, \lambda s\}$. Denote by $d_{k,j}$ and $h_{k,j}$ the k-th entries of d_j and h_j, respectively, for any $d_j \in D$, $h_j \in H$ and $k \in F$. Since A is a Rao-Hamming OA$(N, m, s, 2)$, without loss of generality, assume $N = s^n$ and $m = (s^n - 1)/(s - 1)$. Then $L = (l_1, \dots, l_m)$ is an $n \times m$ matrix based on G.

(i) The number of times that (z_1, z_2, z_3) appears as a row in the subarray $[a_1 \oplus d_1, 0_N \oplus (h_1, h_2)]$ is equal to the number of pairs $(X, x) \in G^n \times F$ such that

$$X^{\mathrm{T}} l_1 + d_{x,1} = z_1, \quad h_{x,1} = z_2, \quad h_{x,2} = z_3. \tag{A.1}$$

For every x satisfying $h_{x,1} = z_2$ and $h_{x,2} = z_3$, there are s^{n-1} solutions for X in (A.1). Since H is an OA$(\lambda s, k, s, t)$ with $t \geqslant 2$, there are λ/s choices for x. So the total number of solutions for (X, x) in (A.1) is λs^{n-2}, implying that

$$[a_1 \oplus d_1, 0_N \oplus (h_1, h_2)]$$

is an OA(3).

(ii) The number of times that (z_1, z_2, z_3) appears as a row in the subarray $[a_i \oplus d_j, a_{i'} \oplus d_{j'}, 0_N \oplus h_1]$ is equal to the number of pairs $(X, x) \in G^n \times F$ such that

$$X^{\mathrm{T}} l_i + d_{x,j} = z_1, \quad X^{\mathrm{T}} l_{i'} + d_{x,j'} = z_2, \quad h_{x,1} = z_3. \tag{A.2}$$

If $i \neq i'$, the first two equations in X in (A.2) are independent. For every x satisfying $h_{x,1} = z_3$, there are s^{n-2} solutions for X in (A.2). Since there are λ choices for x, the total number of solutions for (X, x) in (A.2) is λs^{n-2}, as desired. Next, assume $i = i'$. Then (A.2) indicates

$$h_{x,1} = z_3, \quad d_{x,j} - d_{x,j'} = z_1 - z_2. \tag{A.3}$$

For every x satisfying (A.3), there are s^{n-1} solutions for X in (A.2). Since $s \nmid \lambda$, the number of solutions for x in (A.3) cannot be λs^{-1}, which means that the total number of solutions in (A.2) cannot be λs^{n-2}, implying that

$$[a_i \oplus (d_j, d_{j'}), 0_N \oplus h_1]$$

is not an OA(3).

(iii) The number of times that (z_1, z_2, z_3) appears as a row in the subarray $[a_i \oplus d_j, a_{i'} \oplus (d_1, d_2)]$ is equal to the number of pairs $(X, x) \in G^n \times F$ such that

$$X^{\mathrm{T}} l_i + d_{x,j} = z_1, \quad X^{\mathrm{T}} l_{i'} + d_{x,1} = z_2, \quad X^{\mathrm{T}} l_{i'} + d_{x,2} = z_3. \tag{A.4}$$

If $i \neq i'$, it is seen that (A.4) will not have a solution unless

$$d_{x,2} - d_{x,1} = z_3 - z_2. \tag{A.5}$$

For every x satisfying (A.5), there are s^{n-2} solutions for X in (A.4). Since D is a D$(\lambda s, c, s)$, there are precisely λ values of x satisfying (A.5). Hence, the total number of solutions for (X, x) in (A.4) is λs^{n-2}, as required. Next, assume $i = i'$. We find from (A.4) that

$$d_{x,2} - d_{x,1} = z_3 - z_2, \quad d_{x,1} - d_{x,j} = z_2 - z_1, \tag{A.6}$$

and for every x satisfying (A.6), there are s^{n-1} solutions for X in (A.4). Since $s \nmid \lambda$, the number of solutions for x in (A.6) cannot be λs^{-1}, and the total number of solutions in (A.4) cannot be λs^{n-2}, which is not desired.

(iv) The number of times that (z_1, z_2, z_3) appears as a row in the subarray $[a_1 \oplus d_{j_1}, a_2 \oplus d_{j_2}, a_3 \oplus d_{j_3}]$ is equal to the number of pairs $(X, x) \in G^n \times F$ such that

$$X^{\mathrm{T}} l_1 + d_{x,j_1} = z_1, \quad X^{\mathrm{T}} l_2 + d_{x,j_2} = z_2, \quad X^{\mathrm{T}} l_3 + d_{x,j_3} = z_3. \tag{A.7}$$

Since $\beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 = 0$, (A.7) will not have a solution unless

$$\beta_1 d_{x,j_1} + \beta_2 d_{x,j_2} + \beta_3 d_{x,j_3} = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3, \tag{A.8}$$

and for each x satisfying (A.8), there are s^{n-2} solutions for X in (A.7). Note that (A.8) has λ solutions for any (z_1, z_2, z_3), if and only if $\beta_1 d_{x,j_1} + \beta_2 d_{x,j_2} + \beta_3 d_{x,j_3}$ is an $\mathrm{OA}(\lambda s, 1, s, 1)$.

(v) The number of times that (z_1, z_2, z_3) appears as a row in the subarray $(a_1 \oplus d_{j_1}, a_2 \oplus d_{j_2}, a_3 \oplus d_{j_3})$ is equal to the number of pairs

$$(X, x) \in [\mathrm{GF}(s)]^n \times \{1, \ldots, \lambda s\}$$

such that

$$X^{\mathrm{T}} l_1 + d_{x,j_1} = z_1, \quad X^{\mathrm{T}} l_2 + d_{x,j_2} = z_2, \quad X^{\mathrm{T}} l_3 + d_{x,j_3} = z_3. \tag{A.9}$$

Due to the linearly independence of l_1, l_2 and l_3 over $\mathrm{GF}(s)$, for each $x \in \{1, 2, \ldots, \lambda s\}$, we have three independent equations in X in (A.9), which leads to s^{n-3} solutions. Since there are λs choices for x, there are λs^{n-2} solutions to (A.9). This implies that

$$(a_1 \oplus d_{j_1}, a_2 \oplus d_{j_2}, a_3 \oplus d_{j_3})$$

is an $\mathrm{OA}(\lambda s^{n+1}, 3, s, 3)$. This completes the proof. □

Proof of Theorem 3.2. The numbers of columns in $A^{(1)}, \ldots, A^{(5)}$ are

$$\begin{aligned}
&(s^{n-2} - 1)/(s - 1) - ((s - 1)^{n-2} - 1)/(s - 2), \quad ((s - 1)^{n-2} - 1)/(s - 2), \\
&s^{n-2} - 2(s - 1)^{n-2} + (s - 2)^{n-2}, \quad (s - 1)^{n-2} - (s - 2)^{n-2}
\end{aligned}$$

and $(s-1)^{n-2}$, respectively. Then A in (3.5) does have the required number of factors. Suppose that a_1 and a_2 are any two distinct columns of A, and b_1 is the column corresponding to a_1 in B. In the sequel, we show that (a_1, b_1, a_2) is an OA(3) in all possible cases. Define $G = \mathrm{GF}(s)$.

(i) $a_1 = a_i^{(1)} \oplus d_1 \in A^{(1)} \oplus d_1$, $b_1 = a_{i'}^{(2)} \oplus d_1 \in A^{(2)} \oplus d_1$ satisfying $l_{r,i'}^{(2)}$ equals 1 if $l_{r,i}^{(1)} = \alpha_{s-2}$ and 0 otherwise. Suppose

$$a_2 = a_{i_2}^{(k)} \oplus d_{j_2} \in A.$$

If $a_{i_2}^{(k)} \in \{a_i^{(1)}, a_{i'}^{(2)}\}$, (a_1, b_1, a_2) is an OA(3) by Lemma A.1(iii). If $l_i^{(1)}$, $l_{i'}^{(2)}$ and $l_{i_2}^{(k)}$ are linearly independent over G, (a_1, b_1, a_2) is an OA(3) by Lemma A.1(v). At last, if $l_i^{(1)}$, $l_{i'}^{(2)}$ and $l_{i_2}^{(k)}$ are linearly dependent over G, there are nonzero elements $\beta, \beta' \in G$ satisfying $l_{i_2}^{(k)} = \beta l_i^{(1)} + \beta' l_{i'}^{(2)}$. Since the first nonzero element of $l_{i_2}^{(k)}$ is 1, we have $\beta = 1$ and $l_{1,i_2}^{(k)} = 0$. For any $1 \leqslant j \leqslant n-1$, if $l_{j,i}^{(1)} = \alpha_{s-2}$,

$$l_{j,i_2}^{(k)} = l_{j,i}^{(1)} + \beta' l_{j,i'}^{(2)} \neq \alpha_{s-2};$$

otherwise $l_{j,i_2}^{(k)} = l_{j,i}^{(1)} \neq \alpha_{s-2}$. This means $\alpha_{s-2} \notin l_{i_2}^{(k)}$ and $a_{i_2}^{(k)} \in A^{(2)}$. Since $a_2 = a_{i_2}^{(k)} \oplus d_{j_2} \in A$, we have $2 \leqslant j_2 \leqslant c$. Then $d_{j_2} - \beta d_1 - \beta' d_1 = d_{j_2}$ is an $\mathrm{OA}(\lambda s, 1, s, 1)$, implying that (a_1, b_1, a_2) is an OA(3) by Lemma A.1(iv).

(ii) $a_1 = a_i^{(3)} \oplus d_1 \in A^{(3)} \oplus d_1$, $b_1 = a_{i'}^{(2)} \oplus d_1$ satisfying $l_{r,i'}^{(2)}$, $r > 1$ equals 1 if $l_{r,i}^{(1)} = 1$ and 0 otherwise. Similar to (i), for any $a_2 = a_{i_2}^{(k)} \oplus d_{j_2} \in A$, we only consider

$$l_{i_2}^{(k)} = \beta l_i^{(3)} + \beta' l_{i'}^{(2)}$$

for some nonzero elements $\beta,\beta'\in G$. As the first nonzero entry of $l_{i_2}^{(k)}$ is 1, we have $\beta=1$, $l_{1,i_2}^{(k)}=1$ and $(l_{2,i_2}^{(k)},\ldots,l_{n-1,i_2}^{(k)})$ contains α_{s-2} but not 1. Then $a_{i_2}^{(k)}\in A^{(4)}$. The fact that $a_2=a_{i_2}^{(k)}\oplus d_{j_2}\in A$ indicates $2\leqslant j_2\leqslant c$. By Lemma A.1(iv), (a_1,b_1,a_2) is an OA(3).

(iii) $a_1=a_i^{(5)}\oplus d_1\in A^{(5)}\oplus d_1$ and $b_1=a_i^{(5)}\oplus d_2$. For any $a_2=a_{i_2}^{(k)}\oplus d_{j_2}\in A$, it is not hard to see that $a_{i_2}^{(k)}\neq a_i^{(5)}$. By Lemma A.1(iii), (a_1,b_1,a_2) is an OA(3).

(iv) $a_1=a_i^{(k)}\oplus d_j\in(A^{(1)},A^{(2)})\oplus(d_2,\ldots,d_c)$, $b_1=a_{i'}^{(4)}\oplus d_1$ satisfying $l_{r,i'}^{(4)}$, $r>1$ equals 0 if $l_{r,i}^{(k)}=0$ and α_{s-2} otherwise. Suppose

$$a_2=a_{i_2}^{(k_2)}\oplus d_{j_2}\in A.$$

Similarly, we only need to consider

$$l_{i_2}^{(k_2)}=\beta l_i^{(k)}+\beta' l_{i'}^{(4)}$$

for some nonzero elements $\beta,\beta'\in G$. Since the first nonzero element of $l_{i_2}^{(k_2)}$ is 1, we have $\beta'=1$, $l_{1,i_2}^{k_2}=1$ and $\alpha_{s-2}\notin l_{i_2}^{(k_2)}$. Thus $a_{i_2}^{(k_2)}\in A^{(5)}$. As $a_2=a_{i_2}^{(k_2)}\oplus d_{j_2}\in A$, we obtain $j_2=1$. Note that $d_{j_2}-\beta d_j-\beta' d_1=-\beta d_j$ is an OA$(\lambda s,1,s,1)$. By Lemma A.1(iv), (a_1,b_1,a_2) is an OA(3).

(v) $a_1=a_i^{(k)}\oplus d_j\in(A^{(3)},A^{(4)})\oplus(d_2,\ldots,d_c)$, $b_1=a_{i'}^{(2)}\oplus d_1$ satisfying $l_{r,i'}^{(2)}$ equals 1 if $l_{r,i}^{(k)}=\alpha_{s-2}$ and 0 otherwise. If $a_2=a_{i_2}^{(k_2)}\oplus d_{j_2}\in A$, similar to (i), it suffices to consider that

$$l_{i_2}^{(k_2)}=\beta l_i^{(k)}+\beta' l_{i'}^{(2)}$$

for some nonzero $\beta,\beta'\in G$. Since the first nonzero element of $l_{i_2}^{(k_2)}$ is 1, we have $\beta=1$, $l_{1,i_2}^{(k_2)}=1$ and $\alpha_{s-2}\notin l_{i_2}^{(k_2)}$. This means $a_{i_2}^{(k_2)}\in A^{(5)}$. Then the fact that $a_2=a_{i_2}^{(k_2)}\oplus d_{j_2}\in A$ implies $j_2=1$. By noting that $d_{j_2}-\beta d_j-\beta' d_1=-d_j$ is an OA$(\lambda s,1,s,1)$, we see that (a_1,b_1,a_2) is an OA(3) from Lemma A.1(iv). □

Now we are going to prove Theorem 3.10. We first introduce some definitions in projective geometry and show Propositions A.2–A.4. These propositions are needed for proving Theorem 3.10. Recall that $\mathrm{GF}(s)=\{\alpha_0,\alpha_1,\ldots,\alpha_{s-1}\}$ is a Galois field of order s, where $\alpha_0=0$, $\alpha_1=\alpha$ is a primitive element of GF(s), and $\alpha_2=\alpha^2,\ldots,\alpha_{s-1}=\alpha^{s-1}=1$, and $l_1,\ldots,l_r$ where $r=(s^n-1)/(s-1)$ are all the vectors in $[\mathrm{GF}(s)]^n$ whose first nonzero element is 1. In the language of projective geometry, l_i and $\alpha_k l_i$, $k>0$ represent the same point and the $s+1$ points l_i, l_j $(j\neq i)$, $l_i+\alpha_1 l_j,\ldots,l_i+\alpha_{s-1}l_j$ form a line. It can be verified that among the $s+1$ points of any line, there are s points whose first nonzero elements appear at the same position, and one last point whose first nonzero element appears at a later position. Let l_1 and l_2 be any two of these s points. When restricting the first nonzero element of each point to be 1, the $s+1$ points in the same line can be rewritten as l_1, l_2, $l_i=\alpha_{i-2}l_1+(1-\alpha_{i-2})l_2$, $i=3,\ldots,s$, and $l_{s+1}=(l_{p,1}-l_{p,2})^{-1}(l_1-l_2)$, where $l_{k,i}$ denotes the k-th entry of l_i and p is the smallest k such that $l_{k,1}\neq l_{k,2}$.

Proposition A.2. *For the above points $l_1,\ldots,l_{s+1}$, we have*

(i) $l_{k,i}-l_{k,j}=l_{s+1,k}(l_{p,i}-l_{p,j})$ *for any* $1\leqslant i<j\leqslant s$ *and* $1\leqslant k\leqslant n$;

(ii) $l_{k,s+1}\neq 0$ *if and only if* $\{l_{k,1},\ldots,l_{k,s}\}$ *is a permutation of all the elements in* GF(s);

(iii) $l_{k,s+1}=0$ *if and only if* $l_{k,1}=\cdots=l_{k,s}$.

Proof. Let $\beta=l_{p,1}-l_{p,2}$. Then $\beta\neq 0$ and simple calculation shows $l_1=l_2+\beta l_{s+1}$ and $l_i=l_2+\alpha_{i-2}\beta l_{s+1}$ for $i=3,\ldots,s$. This means

$$l_{k,1}=l_{k,2}+\beta l_{k,s+1}\quad\text{and}\quad l_{k,i}=l_{k,2}+\alpha_{i-2}\beta l_{k,s+1}\tag{A.10}$$

for any $3\leqslant i\leqslant s$ and $1\leqslant k\leqslant n$. Also, note that $l_{p,s+1}=1$. Proposition A.2(i) immediately follows. It is seen that $l_{k,1}=l_{k,2}$ if and only if $l_{k,s+1}=\beta(l_{k,1}-l_{k,2})=0$. We also find from (A.10) that if $l_{k,s+1}=0$, $l_{k,1}=\cdots=l_{k,s}$; otherwise, $l_{k,1},\ldots,l_{k,s}$ are pairwise distinct. Hence (ii) and (iii) in Proposition A.2 follow. □

Proposition A.3. *Suppose that both $l_1=(0,l_{2,1},\dots,l_{n,1})^{\mathrm{T}}$ and $l_2=(1,l_{2,2},\dots,l_{n,2})^{\mathrm{T}}$ do not contain α_{s-2}. Then among the other $s-1$ points in the line formed by l_1 and l_2, there exists a point $l_i=(l_{1,i},\dots,l_{n,i})^{\mathrm{T}}$ satisfying one of the following two properties*:

(i) $l_{1,i}=1$, *and* $(l_{2,i},\dots,l_{n,i})$ *does not contain* α_{s-2};

(ii) $l_{1,i}=1$, *and* $(l_{2,i},\dots,l_{n,i})$ *contains both* 1 *and* α_{s-2}.

Proof. Let $l_i=l_2+\alpha_{i-2}l_1$ for $i=3,\dots,s+1$. Then the points $l_1,\dots,l_{s+1}$ form a line. Denote by $l_{k,i}$ the k-th entry of l_i. Then $l_{1,1}=0$ and $l_{1,i}=1$ for $i=2,\dots,s+1$. Let $u=\{k>1: l_{k,1}\neq 0\}$. Assume first that there exists an integer $k\in u$ such that $l_{k,2}\neq 1$. By Proposition A.2(ii), there must be a point l_i, $i\geqslant 3$ such that $l_{k,i}=1$. This indicates that $(l_{2,i},\dots,l_{n,i})$ contains 1. If $\alpha_{s-2}\in l_i$, l_i satisfies (ii). Otherwise, l_i satisfies (i). Next, we consider the case $l_{k,2}=1$ for all $k\in u$. Select $l_i=l_2+(1-\alpha)l_1$. Since $l_{k,1}\neq\alpha_{s-2}$ for all k, we have $l_{k,i}=l_{k,2}+(1-\alpha)l_{k,1}\neq\alpha_{s-2}$ for all $k\in u$. Note that $l_{k,1}=0$ for all $k\notin u$ and $l_{k,2}\neq\alpha_{s-2}$ for all k. By Proposition A.2(iii), we also have $l_{k,i}=l_{k,2}\neq\alpha_{s-2}$ for all $k\notin u$. Hence l_i satisfies condition (i). This completes the proof. □

Proposition A.4. *Suppose that $(l_{2,1},\dots,l_{n,1})$ does not contain α_{s-2} and $(l_{2,2},\dots,l_{n,2})$ contains α_{s-2} but not 1. Then among the other $s-1$ points in the line formed by $l_1=(1,l_{2,1},\dots,l_{n,1})^{\mathrm{T}}$ and $l_2=(1,l_{2,2},\dots,l_{n,2})^{\mathrm{T}}$, there exists a point $l_i=(l_{1,i},\dots,l_{n,i})^{\mathrm{T}}$ satisfying one of the following three properties*:

(i) $l_{1,i}=0$, *and* $(l_{2,i},\dots,l_{n,i})$ *contains* α_{s-2};

(ii) $l_{1,i}=1$, *and* $(l_{2,i},\dots,l_{n,i})$ *does not contain* α_{s-2};

(iii) $l_{1,i}=1$, *and* $(l_{2,i},\dots,l_{n,i})$ *contains both* 1 *and* α_{s-2}.

Proof. Let $l_i=\alpha_{i-2}l_1+(1-\alpha_{i-2})l_2$ for $i=3,\dots,s$ and $l_{s+1}=(l_{p,1}-l_{p,2})^{-1}(l_1-l_2)$, where $l_{k,i}$ is the k-th entry of l_i and p is the smallest k such that $l_{k,1}\neq l_{k,2}$. Then the points $l_1,\dots,l_{s+1}$ form a line, $l_{1,1}=\cdots=l_{1,s}=1$ and $l_{1,s+1}=0$. Let $u=\{k>1: l_{k,1}\neq l_{k,2}\}$, $u_1=\{k>1: l_{k,1}=1\}$ and $u_2=\{k>1: l_{k,2}=\alpha_{s-2}\}$. Then $p\in u$. Because $\alpha_{s-2}\notin l_1$ and $(l_{2,2},\dots,l_{n,2})$ contains α_{s-2} but not 1, we have $u_1\subset u$, $u_2\subset u$ and $u_2\neq\emptyset$.

Suppose $u\setminus u_1\neq\emptyset$. Take $k_0\in u\setminus u_1$, and then $k_0>1$, $l_{k_0,1}\neq l_{k_0,2}$ and $l_{k_0,1}\neq 1$. It is also known that $l_{k_0,2}\neq 1$. By Proposition A.2(ii), there exists a point l_i, $3\leqslant i\leqslant s$ such that $l_{k_0,i}=1$. Since $l_{1,i}=1$, we have l_i satisfies (iii) if $\alpha_{s-2}\in l_i$ and (ii) otherwise.

Consider the case of $u=u_1$. Note that $l_{1,s+1}=0$. If $\alpha_{s-2}\in l_{s+1}$, l_{s+1} satisfies (i). Otherwise, $l_{k,s+1}\neq\alpha_{s-2}$ for all k. Since $l_{k,1}=1$ for all $k\in u$, by Proposition A.2(i),

$$l_{k,i}-1=l_{k,s+1}(l_{p,i}-1) \tag{A.11}$$

for all $2\leqslant i\leqslant s$ and all $k\in u$. If $p\in u_2$, $l_{p,2}=\alpha_{s-2}$. By noting $(1-\alpha_{s-2})(1-\alpha)\neq 0$, we obtain $\alpha_{s-2}\neq 1+1-\alpha$, which means $l_{p,2}\neq 1+1-\alpha$. If $p\notin u_2$, taking $k_0\in u_2$, we have $k_0>1$, $k_0\neq p$ and $l_{k_0,2}=\alpha_{s-2}$. Because $u_2\subset u=u_1$ and $p\in u$, we have $l_{k_0,1}=l_{p,1}=1$. As $l_{k_0,s+1}\notin\{0,\alpha_{s-2}\}$, we obtain again

$$l_{p,2}=l_{k_0,s+1}^{-1}(l_{k_0,2}-1)+1\neq 1+1-\alpha$$

from (A.11). Note also that $l_{p,1}=1\neq 1+1-\alpha$ and $l_{p,1}\neq l_{p,2}$. By Proposition A.2(ii), there exist a point l_{i_0}, $3\leqslant i_0\leqslant s$ satisfying $l_{p,i_0}=1+1-\alpha$. Since $l_{k,s+1}\neq\alpha_{s-2}$ for all k, it follows from (A.11) that

$$l_{k,i_0}=l_{k,s+1}(l_{p,i_0}-1)+1\neq\alpha_{s-2}$$

for all $k\in u$. When $k\notin u$, $l_{k,1}=l_{k,2}$ implies $l_{k,s+1}=0$. By Proposition A.2(iii), we find

$$l_{k,i_0}=l_{k,1}\neq\alpha_{s-2}$$

for all $k\notin u$. Hence, l_{i_0} satisfies (ii). This completes the proof. □

Now we turn to the proof of Theorem 3.10. Use the notations in Subsection 3.1. Recall that the constructed design A in Theorem 3.2 is a subarray of the base orthogonal array D_1 in (3.4). Combining (3.3)–(3.5), we see that the complement design of A is

$$\bar{A}=D_1\setminus A=(0_{s^{n-1}}\oplus H, A^{(2)}\oplus d_1, A^{(4)}\oplus d_1, A^{(5)}\oplus(d_2,\dots,d_c)). \tag{A.12}$$

So we correspondingly split the proof of Theorem 3.10 into the following Lemmas A.5–A.8.

Lemma A.5. *For the constructed design A in Theorem 3.2, the column $0_{s^{n-1}} \oplus H$ cannot be added to A.*

Proof. Let $a = 0_{s^{n-1}} \oplus H$. It suffices to prove that for each column $b \in \bar{A} \setminus \{a\}$, there exists a column $a_1 \in A$ such that (a, b, a_1) is not an OA(3). We have the following three cases:

- If $b = a_i^{(2)} \oplus d_1 \in A^{(2)} \oplus d_1$, take $a_1 = a_i^{(2)} \oplus d_2$.
- If $b = a_i^{(4)} \oplus d_1 \in A^{(4)} \oplus d_1$, take $a_1 = a_i^{(4)} \oplus d_2$.
- If $b = a_i^{(5)} \oplus d_j \in A^{(5)} \oplus (d_2, \dots, d_c)$, take $a_1 = a_i^{(5)} \oplus d_1$.

For all the cases, $a_1 \in A$ by (3.5) and (a, b, a_1) is not an OA(3) by Lemma A.1(ii). This completes the proof. □

Lemma A.6. *For the constructed design A in Theorem 3.2, no column in $A^{(2)} \oplus d_1$ can be added to A.*

Proof. It suffices to prove that for any column $a = a_i^{(2)} \oplus d_1$ and any column $b \in \bar{A} \setminus \{a\}$, there exists a column $a_1 \in A$ such that (a, b, a_1) is not an OA(3). Since $a_i^{(2)} \in A^{(2)}$, we have $l_{1,i}^{(2)} = 0$ and $\alpha_{s-2} \notin l_i^{(2)}$.

If $b = 0_{s^{n-1}} \oplus H$, take $a_1 = a_i^{(2)} \oplus d_2$. Then $a_1 \in A$ by (6). By Lemma A.1(ii), (a, b, a_1) is not an OA(3).

If $b = a_{i'}^{(2)} \oplus d_1 \in A^{(2)} \oplus d_1$, we have $l_{i'}^{(2)} \neq l_i^{(2)}$, $l_{1,i'}^{(2)} = 0$ and $\alpha_{s-2} \notin l_{i'}^{(2)}$. It is seen from Proposition A.2(ii) that among the $s+1$ points in any line, there exists at least one point containing α_{s-2}. Let $l_{i_1}^{(q)}$ be such a point in the line formed by $l_i^{(2)}$ and $l_{i'}^{(2)}$. Since $l_{1,i}^{(2)} = l_{1,i'}^{(2)} = 0$, we also have $l_{1,i_1}^{(q)} = 0$, implying $q = 1$. Take $a_1 = a_{i_1}^{(q)} \oplus d_1 \in A^{(1)} \oplus d_1 \subset A$. By Lemma A.1(iv), (a, b, a_1) is not an OA(3).

If $b = a_{i'}^{(4)} \oplus d_1 \in A^{(4)} \oplus d_1$, we have $l_{1,i'}^{(4)} = 1$ and $(l_{2,i'}^{(4)}, \dots, l_{n-1,i'}^{(4)})$ contains α_{s-2} but not 1. There must be an integer $k > 1$ such that $l_{k,i}^{(2)} = 1$. Take $\beta = 1 - l_{k,i'}^{(4)}$ and $l_{i_1}^{(q)} = l_{i'}^{(4)} + \beta l_i^{(2)}$. Then $\beta \neq 0$ and $l_{1,i_1}^{(q)} = l_{k,i_1}^{(q)} = 1$. So $q = 3$ if $\alpha_{s-2} \in l_{i_1}^{(q)}$ and $q = 5$ otherwise. Take

$$a_1 = a_{i_1}^{(q)} \oplus d_1 \in (A^{(3)}, A^{(5)}) \oplus d_1 \subset A.$$

By Lemma A.1(iv), (a, b, a_1) is not an OA(3).

If $b = a_{i'}^{(5)} \oplus d_{j'} \in A^{(5)} \oplus (d_2, \dots, d_c)$, we have $j' > 1$, $l_{1,i'}^{(5)} = 1$ and $\alpha_{s-2} \notin l_{i'}^{(5)}$. There must be an integer $k > 1$ such that $l_{k,i}^{(2)} = 1$. Let $\beta = \alpha_{s-2} - l_{k,i'}^{(5)}$ and $l_{i_1}^{(q)} = l_{i'}^{(5)} + \beta l_i^{(2)}$. Then $\beta \neq 0$, $l_{1,i_1}^{(q)} = 1$, $l_{k,i_1}^{(q)} = \alpha_{s-2}$ and $q \in \{3, 4\}$. Let

$$a_1 = a_{i_1}^{(q)} \oplus d_{j'} \in (A^{(3)}, A^{(4)}) \oplus (d_2, \dots, d_c) \subset A.$$

By Lemma A.1(iv), (a, b, a_1) is not an OA(3). □

Lemma A.7. *For the constructed design A in Theorem 3.2, no column in $A^{(4)} \oplus d_1$ can be added to A, provided that D_0 possesses the property δ.*

Proof. It suffices to prove that for any column $a = a_i^{(4)} \oplus d_1 \in A^{(4)} \oplus d_1$ and any column $b \in \bar{A} \setminus \{a\}$, there exists a column $a_1 \in A$ such that (a, b, a_1) is not an OA(3). Since $a_i^{(4)} \in A^{(4)}$, we have $l_{1,i}^{(4)} = 1$ and $(l_{2,i}^{(4)}, \dots, l_{n-1,i}^{(4)})$ contains α_{s-2} but not 1.

If $b = 0_{s^{n-1}} \oplus H$, take

$$a_1 = a_i^{(4)} \oplus d_2 \in A^{(2)} \oplus d_2 \subset A.$$

By Lemma A.1(ii), (a, b, a_1) is not an OA(3).

If $b \in A^{(2)} \oplus d_1$, as in the proof of Lemma A.6, we have shown that for any two columns $a \in A^{(2)} \oplus d_1$ and $b \in A^{(4)} \oplus d_1$, there exists a column $a_1 \in A$ such that (a, b, a_1) is not an OA(3). So the proof of this case is omitted.

If $b = a_{i'}^{(4)} \oplus d_1 \in A^{(4)} \oplus d_1$, we have $i' \neq i$, $l_{1,i'}^{(4)} = 1$, and $(l_{2,i'}^{(4)}, \dots, l_{n-1,i'}^{(4)})$ contains α_{s-2} but not 1. There must be an integer $k > 1$ so that $l_{k,i}^{(4)} \neq l_{k,i'}^{(4)}$. Since $l_{k,i}^{(4)} \neq 1$ and $l_{k,i'}^{(4)} \neq 1$, by Proposition A.2(ii), there exists a point $l_{i_1}^{(q)} = \beta l_i^{(4)} + (1-\beta) l_{i'}^{(4)}$ such that $l_{k,i_1}^{(q)} = 1$. As $l_{1,i_1}^{(q)} = \beta l_{1,i}^{(4)} + (1-\beta) l_{1,i'}^{(4)} = 1$, we have $q = 3$ if $\alpha_{s-2} \in l_{i_1}^{(q)}$ and $q = 5$ otherwise. Take

$$a_1 = a_{i_1}^{(q)} \oplus d_1 \in (A^{(3)}, A^{(5)}) \oplus d_1 \subset A.$$

By Lemma A.1(iv), (a, b, a_1) is not an OA(3).

If $b = a_{i'}^{(5)} \oplus d_{j'} \in A^{(5)} \oplus (d_2, \ldots, d_c)$, we have $j' > 1$, $l_{1,i'}^{(5)} = 1$ and $\alpha_{s-2} \notin l_{i'}^{(5)}$. Let

$$l_{i_1}^{(q)} = \beta(l_i^{(4)} - l_{i'}^{(5)}),$$

where $\beta \in \mathrm{GF}(s)$ is the element such that the first nonzero element of $l_{i_1}^{(q)}$ is 1. Then $l_{1,i_1}^{(q)} = 0$ and $q \in \{1, 2\}$. Since D_0 has the property δ, take j_1 so that $d_{j_1} + \beta d_{j'}$ is not an OA$(\lambda s, 1, s, 1)$. As $d_1 = 0_{\lambda s}$ and $\beta d_{j'}$ is an OA$(\lambda s, 1, s, 1)$, we have $j_1 > 1$. Take

$$a_1 = a_{i_1}^{(q)} \oplus d_{j_1} \in (A^{(1)}, A^{(2)}) \oplus d_{j_1} \subset A.$$

By Lemma A.1(iv), (a, b, a_1) is not an OA(3). □

Lemma A.8. *For the constructed design A in Theorem 3.2, no column in $A^{(5)} \oplus (d_2, \ldots, d_c)$ can be added to A, provided that D_0 possesses the property δ.*

Proof. Suppose

$$a = a_{i_1}^{(5)} \oplus d_{j_1} \in A^{(5)} \oplus (d_2, \ldots, d_c).$$

Then we have $j_1 > 1$, $l_{1,i_1}^{(5)} = 1$ and $\alpha_{s-2} \notin l_{i_1}^{(5)}$. Take

$$a_1 = a_{i_1}^{(5)} \oplus d_1 \in A^{(5)} \oplus d_1 \subset A.$$

We show that a and a_1 cannot be included by A at the same time. It suffices to prove that for any column $b_1 \in \bar{A} \setminus \{a\}$, (a_1, b_1, a) is not an OA(3), or there exists a column $a_2 \in A \setminus \{a_1\}$, such that (a_1, b_1, a_2) is not an OA(3).

If $b_1 = 0_{s^{n-1}} \oplus H$, (a_1, b_1, a) is not an OA(3) by Lemma A.1(ii).

If $b_1 = a_i^{(2)} \oplus d_1 \in A^{(2)} \oplus d_1$, we have $l_{1,i}^{(2)} = 0$ and $\alpha_{s-2} \notin l_i^{(2)}$. From Proposition A.3, among the other $s-1$ points in the line formed by $l_{i_1}^{(5)}$ and $l_i^{(2)}$, there must be a point, i.e., $l_{i_2}^{(q)}$, satisfying (i) or (ii) in Proposition A.3. More precisely, $q = 5$ if $l_{i_2}^{(q)}$ satisfies Proposition A.3(i) and $q = 3$ if $l_{i_2}^{(q)}$ satisfies Proposition A.3(ii). Take

$$a_2 = a_{i_2}^{(q)} \oplus d_1 \in (A^{(3)}, A^{(5)}) \oplus d_1 \subset A.$$

By Lemma A.1(iv), (a_1, b_1, a_2) is not an OA(3).

If $b_1 = a_i^{(4)} \oplus d_1 \in A^{(4)} \oplus d_1$, we have $l_{1,i}^{(4)} = 1$, and $(l_{2,i}^{(4)}, \ldots, l_{n-1,i}^{(4)})$ contains α_{s-2} but not 1. From Proposition A.4, among the other $s-1$ points in the line formed by $l_{i_1}^{(5)}$ and $l_i^{(4)}$, there must be a point, i.e., $l_{i_2}^{(q)}$, satisfying (i), (ii) or (iii) in Proposition A.4, indicating $q = 1$, $q = 5$ or $q = 3$, respectively. Take

$$a_2 = a_{i_2}^{(q)} \oplus d_1 \in (A^{(1)}, A^{(3)}, A^{(5)}) \oplus d_1 \subset A.$$

By Lemma A.1(iv), (a_1, b_1, a_2) is not an OA(3).

If $b_1 = a_i^{(5)} \oplus d_j \in A^{(5)} \oplus (d_2, \ldots, d_c)$, we have $j > 1$, $l_{1,i}^{(5)} = 1$ and $\alpha_{s-2} \notin l_i^{(5)}$. If $i = i_1$, (a_1, b_1, a) is not an OA(3) by Lemma A.1(iii). Otherwise, let $l_{i_2}^{(q)} = \beta(l_{i_1} - l_i)$, where $\beta \in \mathrm{GF}(s)$ is the element such that the first nonzero element of $l_{i_2}^{(q)}$ is 1. Then $l_{1,i_2} = 0$ and $q \in \{1, 2\}$. As D_0 has the property δ, take j_2 so that $d_{j_2} + \beta d_j$ is not an OA$(\lambda s, 1, s, 1)$. Because $d_1 = 0_{\lambda s}$ and βd_j is an OA$(\lambda s, 1, s, 1)$, we have $j_2 > 1$. Take

$$a_2 = a_{i_2}^{(q)} \oplus d_{j_2} \in (A^{(1)}, A^{(2)}) \oplus d_{j_2} \subset A.$$

By Lemma A.1(iv), (a_1, b_1, a_2) is not an OA(3). □

Proof of Theorem 3.10. Combining Lemmas A.5–A.8, we see that Theorem 3.10 immediately follows. □

Proof of Corollary 3.3. For any prime power s, [12, Theorem 6.6, Corollary 6.39 and Theorem 6.63] illustrates the existence of the difference schemes $\mathrm{D}(s,s,s)$, $\mathrm{D}(2s,2s,s)$ and $\mathrm{D}(4s,4s,s)$, respectively, based on $\mathrm{GF}(s)$. [12, Theorem 6.64] shows that a difference scheme $\mathrm{D}((s-1)^k s,(s-1)^k s,s)$ exists when $s-1$ is a prime power and k is a positive integer. □

Proof of Theorem 4.2. Because A and B are subarrays of H_q without repeated columns, and the matrices in (4.7) and (4.10) have $2cs^{n-2k-1}-1$ and $c(s-3)s^{n-2k-1}+1$ columns, respectively, both A and B are $\mathrm{OA}(2)$ with the required number of columns. Let a_1' and a_2' be two distinct columns of A, and b_1' be the column corresponding to a_1' in B. In the sequel, we show that (a_1',b_1',a_2') is an $\mathrm{OA}(3)$ in all possible cases. Denote by h_j the j-th columns of H_0, where $1\leqslant j\leqslant c(s^{n-2q-1}-1)/(s-1)+1$.

(i) Suppose that $a_1'=0_{s^{2k-2}}\oplus a_1\oplus d_j^{(n-2k)}$, where $1\leqslant k\leqslant q$ and $2\leqslant j\leqslant cs^{n-2k-1}$. Then

$$b_1'=0_{s^{2k-2}}\oplus a_4\oplus d_j^{(n-2k)}.$$

If $a_2'=0_{s^{2q}}\oplus h_{j_2}$, from Lemma A.1(ii), the array

$$V_0=[(a_1,a_4)\oplus d_j^{(n-2k)},0_{s^{2q-2k+2}}\oplus h_{j_2}]$$

is an $\mathrm{OA}(3)$, implying that $(a_1',b_1',a_2')=0_{s^{2k-2}}\oplus V_0$ is also an $\mathrm{OA}(3)$. Next, assume

$$a_2'=0_{s^{2k_2-2}}\oplus a_{i_2}\oplus d_{j_2}^{(n-2k_2)}.$$

If $1\leqslant k_2<k$, Lemma A.1(i) implies

$$[a_{i_2}\oplus d_{j_2}^{(n-2k_2)},0_{s^{2k-2k_2}}\oplus(a_1,a_4)\oplus d_j^{(n-2k)}]$$

is an $\mathrm{OA}(3)$. If $k<k_2\leqslant q$, Lemma A.1(ii) implies

$$[(a_1,a_4)\oplus d_j^{(n-2k)},0_{s^{2k_2-2k}}\oplus a_{i_2}\oplus d_{j_2}^{(n-2k_2)}]$$

is an $\mathrm{OA}(3)$. So for both cases, (a_1',b_1',a_2') is an $\mathrm{OA}(3)$. Finally, we consider $k_2=k$. It suffices to prove that

$$U=[(a_1,a_4)\oplus d_j^{(n-2k)},a_{i_2}\oplus d_{j_2}^{(n-2k)}]$$

is an $\mathrm{OA}(3)$. If $a_{i_2}\in\{a_1,a_4\}$, Lemma A.1(iii) implies U is an $\mathrm{OA}(3)$. If $a_{i_2}=a_2$, we have $j_2>1$. It is seen that $l_2=l_4-l_1$ and

$$d_{j_2}^{(n-2k)}-d_j^{(n-2k)}+d_j^{(n-2k)}=d_{j_2}^{(n-2k)}$$

is an $\mathrm{OA}(\lambda s^{n-2k},1,s,1)$. If $a_{i_2}=a_3$, we have $j_2=1$. It is seen that $l_3=l_4-\alpha_{s-2}l_1$ and

$$d_{j_2}^{(n-2k)}-d_j^{(n-2k)}+\alpha_{s-2}d_j^{(n-2k)}=(\alpha_{s-2}-1)d_j^{(n-2k)}$$

is an $\mathrm{OA}(\lambda s^{n-2k},1,s,1)$. By Lemma A.1(iv), U is an $\mathrm{OA}(3)$ for both cases.

(ii) If $a_1'=0_{s^{2k-2}}\oplus a_2\oplus d_j^{(n-2k)}$ where $1\leqslant k\leqslant q$ and $2\leqslant j\leqslant cs^{n-k-1}$, we have

$$b_1'=0_{s^{2k-2}}\oplus a_3\oplus d_j^{(n-2k)}.$$

Similar to (i), it suffices to consider that $a_2'=0_{s^{2k-2}}\oplus a_{i_2}\oplus d_{j_2}^{(n-2k)}$ with $a_{i_2}\notin\{a_2,a_3\}$. This means $a_{i_2}=a_1$ and $j_2>1$. Note that $l_1=(l_2-l_3)/(\alpha_{s-2}-1)$ and

$$d_{j_2}-(d_j-d_j)/(\alpha_{s-2}-1)=d_{j_2}$$

is an $\mathrm{OA}(\lambda s^{n-2k},1,s,1)$. By Lemma A.1(iv), $[(a_2,a_3)\oplus d_j^{(n-2k)},a_{i_2}\oplus d_{j_2}^{(n-2k)}]$ is an $\mathrm{OA}(3)$, implying that (a_1',b_1',a_2') is also an $\mathrm{OA}(3)$.

(iii) If $a_1' = 0_{s^{2k-2}} \oplus a_3 \oplus d_1^{(n-2k)}$, we have $b_1' = 0_{s^{2k-2}} \oplus a_2 \oplus d_1^{(n-2k)}$. Similar to (i), it suffices to consider that $a_2' = 0_{s^{2k-2}} \oplus a_{i_2} \oplus d_{j_2}^{(n-2k)}$ with $a_{i_2} \notin \{a_2, a_3\}$. Then $a_{i_2} = a_1$ and $j_2 > 1$. Similar to (ii), (a_1', b_1', a_2') is an OA(3).

(iv) Assume $a_1' = 0_{s^{2q}} \oplus h_j$. Then $b_1' = 0_{s^{2k-2}} \oplus a_{i_1} \oplus d_{j_1}^{(n-2k)}$ satisfying $a_{i_1} \notin \{a_1, a_2, a_3\}$. If $a_2' = 0_{s^{2q}} \oplus h_{j_2}$ where $j_2 \neq j$, we have that $[a_{i_1} \oplus d_{j_1}^{(n-2k)}, 0_{s^{2q-2k+2}} \oplus (h_j, h_{j_2})]$ is an OA(3) by Lemma A.1(i), implying that (a_1', b_1', a_2') is also an OA(3). Assume $a_2' = 0_{s^{2k_2-2}} \oplus a_{i_2} \oplus d_{j_2}^{(n-2k_2)}$. Then we must have $a_{i_2} \in \{a_1, a_2, a_3\}$, indicating $a_{i_2} \neq a_{i_1}$. If $k_2 = k$, Lemma A.1(ii) implies that

$$[0_{s^{2q-2k+2}} \oplus h_j, a_{i_1} \oplus d_{j_1}^{(n-2k)}, a_{i_2} \oplus d_{j_2}^{(n-2k)}]$$

is an OA(3). So (a_1', b_1', a_2') is an OA(3). If $1 \leqslant k_2 < k$, since $(a_{i_1} \oplus d_{j_1}^{(n-2k)}, 0_{s^{2q-2k+2}} \oplus h_j)$ is an OA(2),

$$[a_{i_2} \oplus d_{j_2}^{(n-2k_2)}, 0_{s^{2k-2k_2}} \oplus (a_{i_1} \oplus d_{j_1}^{(n-2k)}, 0_{s^{2q-2k+2}} \oplus h_j)]$$

is an OA(3) by Lemma A.1(i). Hence, (a_1', b_1', a_2') is an OA(3). For $k < k_2 \leqslant q$, the proof is similar and thus omitted. □

Proof of Corollary 4.3. It is seen that $q^* = q_0 = 1$ for $3 \leqslant n \leqslant 4$. So it suffices to consider the case of $n \geqslant 5$.

(i) For $s = 3$, the formula of m_{q_0} follows directly from (4.9) and (4.11). For any $2 \leqslant q \leqslant q_0$,

$$m_q - m_{q-1} = 2c \cdot 3^{n-2q-1} - 1 + r_q - r_{q-1}.$$

Note that $n \geqslant 2q+1$ if n is odd and $n \geqslant 2q+2$ otherwise. Since $n \leqslant 4(c+1)$, it follows that $r_q \geqslant 1$, $r_{q-1} = q-1$ and $q \leqslant 2c+1$. So $m_q - m_{q-1} \geqslant 2c - q + 1 \geqslant 0$, implying $q^* = q_0$.

(ii) For $s > 3$, we have $r_1 = c(s^{n-3}-1)/(s-1) + 1$ and

$$\begin{aligned} m_q &= (2cs^{n-3} - 1) + (2cs^{n-5} - 1) + \cdots + (2cs^{n-2q-1} - 1) + r_q \\ &< (2cs^{n-3} - 1) + (cs^{n-4} + cs^{n-5} + \cdots + cs^{n-2q-1}) + c(s^{n-2q-1} - 1)/(s-1) + 1 \\ &= (2cs^{n-3} - 1) + c(s^{n-3} - 1)/(s-1) + 1 = m_1 \end{aligned}$$

for any $2 \leqslant q \leqslant q_0$. Hence $q^* = 1$. □

Proof of Theorem 5.1. Note that when the s^2 levels are collapsed into s levels by $\lfloor x/s \rfloor$ for $x \in \mathbb{Z}_{s^2}$, the design D becomes $\phi(A)$. So $\pi(D)$ is actually the proportion of strength-three subarrays in all three-column subarrays of A. From the definition of A in (3.5), when $n = 3$, we have

$$A = [(A^{(2)}, A^{(4)}) \oplus (d_2, \ldots, d_c), A^{(5)} \oplus d_1], \tag{A.13}$$

where $A^{(2)}$ and $A^{(4)}$ are s^2-vectors generated by $(0,1)^{\mathrm{T}}$ and $(1, \alpha_{s-2})^{\mathrm{T}}$, respectively, and $A^{(5)}$ is an $s^2 \times (s-1)$ matrix whose columns are generated by vectors $(1, \beta)^{\mathrm{T}}$ with $\beta \neq \alpha_{s-2}$. Without loss of generality, assume that

$$U = (a_1 \oplus d_{j_1}, a_2 \oplus d_{j_2}, a_3 \oplus d_{j_3}) \tag{A.14}$$

is a strength-three subarray of A. Denote by l_1, l_2 and l_3 the generator columns of a_1, a_2 and a_3, respectively.

(i) Consider first $a_2 = a_3$. Without loss of generality, assume $j_2 \leqslant j_3$. Then (A.13) implies $a_2, a_3 \in (A^{(2)}, A^{(4)})$ and $1 < j_2 < j_3 \leqslant c$. By Lemma A.1(iii), U is an OA(3) if and only if $a_1 \neq a_2$. Only two cases need to be considered: (a) $a_1 \in (A^{(2)}, A^{(5)})$ and $a_2 = a_3 = A^{(4)}$; (b) $a_1 \in (A^{(4)}, A^{(5)})$ and $a_2 = a_3 = A^{(2)}$. For each case, there are $(s+c-2)$ and $\binom{c-1}{2}$ choices for (a_1, j_1) and (j_2, j_3), respectively. So there are $(s+c-2)(c-1)(c-2)$ choices for U.

(ii) Assume that a_1, a_2 and a_3 are distinct and $j_1 = j_2 = 1$. Then (A.13) implies $a_1, a_2 \in A^{(5)}$. Note that l_1, l_2 and l_3 are linearly dependent when $n = 3$. By Lemma A.1(iv), U is an OA(3) if and only if $j_3 > 1$. Hence $a_3 \in (A^{(2)}, A^{(4)})$. Then there are $\binom{s-1}{2}$, 2 and $c-1$ choices for (a_1, a_2), a_3 and j_3, respectively. So there are $(s-1)(s-2)(c-1)$ choices for U.

(iii) Assume that a_1, a_2 and a_3 are distinct and $1 = j_1 < j_2 \leqslant j_3 \leqslant c$. Then $a_1 \in A^{(5)}$, $a_2, a_3 \in (A^{(2)}, A^{(4)})$ and $l_1 = (1, \beta')^{\mathrm{T}}$ with $\beta' \neq \alpha_{s-2}$. Note that

$$l_3 = \beta_1 l_1 + \beta_2 l_2 \tag{A.15}$$

for some nonzero $\beta_1, \beta_2 \in \mathrm{GF}(s)$. By Lemma A.1(iv), U is an OA(3) if and only if $d_{j_3} - \beta_2 d_{j_2}$ is an OA$(\lambda s, 1, s, 1)$.

If $j_2 = j_3$, we must have $\beta_2 \neq 1$. There are $c-1$ and $s-2$ choices for j_2 and β_2, respectively. So there are $(c-1)(s-2)$ choices for U.

Next, we consider the case where $j_2 < j_3$ and $\beta_2 = 1$. In this case, $d_{j_3} - \beta_2 d_{j_2}$ is an OA$(\lambda s, 1, s, 1)$. If $a_2 \in A^{(2)}$, we have $a_3 \in A^{(4)}$, $l_2 = (0, 1)^{\mathrm{T}}$ and $l_3 = (1, \alpha_{s-2})^{\mathrm{T}}$. From (A.15), we have

$$1 = \beta_1 \quad \text{and} \quad \alpha_{s-2} = \beta_1 \beta' + 1. \tag{A.16}$$

So $\beta' = \alpha_{s-2} - 1 \neq \alpha_{s-2}$, as desired. There are $\binom{c-1}{2}$ choices for (j_2, j_3). So there are $\binom{c-1}{2}$ choices for U. If $a_2 \in A^{(4)}$, we have $a_3 \in A^{(2)}$, $l_2 = (1, \alpha_{s-2})^{\mathrm{T}}$ and $l_3 = (0, 1)^{\mathrm{T}}$. From (A.15), we have

$$0 = \beta_1 + 1 \quad \text{and} \quad 1 = \beta_1 \beta' + \alpha_{s-2}. \tag{A.17}$$

It is obtained again that $\beta' = \alpha_{s-2} - 1 \neq \alpha_{s-2}$ and there are $\binom{c-1}{2}$ choices for U. So there are $(c-1)(c-2)$ choices for U in (A.14) when $1 = j_1 < j_2 < j_3 \leqslant c$ and $\beta_2 = 1$.

At last, we consider $j_2 < j_3$ and $\beta_2 \neq 1$. Then $\beta_2 \in \mathrm{GF}(s) \setminus \{0, 1\}$. If $a_2 \in A^{(2)}$, we have $a_3 \in A^{(4)}$, $l_2 = (0, 1)^{\mathrm{T}}$ and $l_3 = (1, \alpha_{s-2})^{\mathrm{T}}$. From (A.15), we have

$$1 = \beta_1 \quad \text{and} \quad \alpha_{s-2} = \beta_1 \beta' + \beta_2. \tag{A.18}$$

This indicates $\beta' = \alpha_{s-2} - \beta_2 \neq \alpha_{s-2}$, as required. So for each $\beta_2 \notin \{0, 1\}$, U is an OA(3) provided that $d_{j_3} - \beta_2 d_{j_2}$ is an OA$(\lambda s, 1, s, 1)$. This means that the number of choices for U in (A.14) is ω_0. If $a_2 \in A^{(4)}$, we have $a_3 \in A^{(2)}$, $l_2 = (1, \alpha_{s-2})^{\mathrm{T}}$ and $l_3 = (0, 1)^{\mathrm{T}}$. Then (A.15) implies

$$0 = \beta_1 + \beta_2 \quad \text{and} \quad 1 = \beta_1 \beta' + \beta_2 \alpha_{s-2}. \tag{A.19}$$

Then $\beta' = \alpha_{s-2} - \beta_2^{-1} \neq \alpha_{s-2}$, as desired. So the number of choices for U is ω_0 again.

From (A.13), it is impossible that j_1, j_2 and j_3 all exceed 1. Note that A in (A.13) has $2c + s - 3$ factors. Combining (i)–(iii), we see that the desired result follows. $\square$

Proof of Theorem 5.2. For $n = 3$, the fact that $1 \leqslant q \leqslant \lfloor (n-1)/2 \rfloor$ implies $q = 1$. Similar to the proof of Theorem 5.1, $\pi(D)$ is actually the proportion of strength-three subarrays in all three-column subarrays of A. Let $U = (u_1, u_2, u_3)$ be a strength-three three-column subarray of A. When $n = 3$, we have

$$A = [(a_1, a_2) \oplus (d_2^{(1)}, \ldots, d_c^{(1)}), a_3 \oplus d_1^{(1)}, 0_{\lambda s^2} \oplus v]. \tag{A.20}$$

It is seen that A has $2c$ factors.

Consider first that $u_3 = 0_{\lambda s^2} \oplus v$. Then we can define $u_1 = a_i \oplus d_j^{(1)}$ and $u_2 = a_{i'} \oplus d_{j'}^{(1)}$. By Lemma A.1(ii), U is an OA(3) if and only if $a_i \neq a_{i'}$. Two cases need to be considered: (a) $a_i = a_1$ and $a_{i'} = a_2$; (b) $a_i \in \{a_1, a_2\}$ and $a_{i'} = a_3$. For the two cases, there are altogether $(c^2 - 1)$ choices for U to be an OA(3).

Next, assume $u_1 = a_i \oplus d_{j_1}^{(1)}$, $u_2 = a_i \oplus d_{j_2}^{(1)}$ and $u_3 = a_{i'} \oplus d_{j_3}^{(1)}$. By Lemma A.1(iii), U is an OA(3) if and only if $a_i \neq a_{i'}$. If $a_i \in \{a_1, a_2\}$ and $a_{i'} = a_3$, there are $(c-1)(c-2)$ choices for U. If $a_i, a_i' \in \{a_1, a_2\}$, there are $(c-1)^2(c-2)$ choices for U. So the total number of choices is $c(c-1)(c-2)$.

At last, we consider $u_1 = a_1 \oplus d_{j_1}^{(1)}$, $u_2 = a_2 \oplus d_{j_2}^{(1)}$ and $u_3 = a_3 \oplus d_1^{(1)}$. Note that $l_3 = (1 - \alpha_{s-2}) l_1 + l_2$. By Lemma A.1(iv), U is an OA(3) if and only if $d_{j_2}^{(1)} - (\alpha_{s-2} - 1) d_{j_1}^{(1)}$ is an OA$(\lambda s, 1, s, 1)$. There are w_1 choices for (j_1, j_2). So there are w_1 choices for U.

Combining the above facts, we see that the total number of choices for U is $(c-1)(c^2 - c + 1) + \omega_1$. So the desired result in (5.2) follows. When $s = 3$, $\alpha_{s-2} = 2$. Then $d_{j'}^{(1)} - d_j^{(1)}$ is an OA$(\lambda s, 1, s, 1)$ if and only if $j \neq j'$. Hence $\omega_1 = (c-1)(c-2)$ and $\pi(D) = (3c^2 - 3)/(4c^2 - 2c)$. $\square$

Appendix B Small difference schemes used for constructing strong orthogonal arrays

Table B.1 displays the seven small order difference schemes utilized in this paper. The scheme D(s,s,s) is the $s \times s$ multiplication table of the field GF(s) for $s = 4$ and $s = 5$. The schemes D$(6,6,3)$ and D$(10,10,5)$ are obtained from Tables 6.37 and 6.35 in [12], respectively, while D$(8,8,4)$ is derived from [12, Theorem 6.6]. The scheme D$(12,12,3)$ is sourced from [23], whereas D$(12,12,4)$ is sourced from [22], with the original symbols $00, 01, 10$ and 11 converted to $0, 1, x$ and $1+x$.

Table B.1 Difference schemes D$(4,4,4)$, D$(5,5,5)$, D$(6,6,3)$, D$(8,8,4)$, D$(10,10,5)$, D$(12,12,3)$ and D$(12,12,4)$

D(4,4,4)			
0	0	0	0
0	$1+x$	1	x
0	1	x	$1+x$
0	x	$1+x$	1

D(5,5,5)				
0	0	0	0	0
0	1	2	3	4
0	2	4	1	3
0	3	1	4	2
0	4	3	2	1

D(6,6,3)					
0	0	0	0	0	0
0	1	2	1	2	0
0	2	1	1	0	2
0	2	2	0	1	1
0	0	1	2	2	1
0	1	0	2	1	2

D(8,8,4)							
0	0	0	0	0	0	0	0
0	$1+x$	$1+x$	1	0	x	1	x
0	$1+x$	1	0	x	1	x	$1+x$
0	1	0	x	1	x	$1+x$	$1+x$
0	0	x	1	x	$1+x$	$1+x$	1
0	x	1	x	$1+x$	$1+x$	1	0
0	1	x	$1+x$	$1+x$	1	0	x
0	x	$1+x$	$1+x$	1	0	x	1

D(10,10,5)									
0	0	0	0	0	0	0	0	0	0
0	4	3	1	2	1	0	4	2	3
0	3	1	2	4	4	2	0	1	3
0	1	2	4	3	1	2	3	0	4
0	2	4	3	1	4	1	3	2	0
0	2	3	2	3	0	4	1	4	1
0	1	1	3	0	2	4	4	3	2
0	0	4	4	2	3	3	1	1	2
0	3	0	1	1	2	3	2	4	4
0	4	2	0	4	3	1	2	3	1

D(12,12,3)											
0	0	0	1	1	0	0	1	0	2	2	0
0	0	0	0	2	0	2	0	2	0	0	1
0	0	1	0	0	2	1	2	0	0	1	0
0	0	2	2	0	1	0	0	1	1	0	0
0	1	2	2	0	0	1	1	2	0	2	2
0	1	2	1	2	1	2	2	2	2	1	0
0	1	0	0	2	2	0	2	1	1	2	2
0	1	1	2	1	2	2	0	0	2	0	2
0	2	1	2	1	0	0	2	2	1	1	1
0	2	1	0	0	1	2	1	1	2	2	1
0	2	2	1	2	2	1	1	0	1	0	1
0	2	0	1	1	1	1	0	1	0	1	2

D(12,12,4)											
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	$1+x$	$1+x$	$1+x$	x	x	x
0	0	0	$1+x$	$1+x$	$1+x$	x	x	x	1	1	1
0	$1+x$	1	x	1	$1+x$	1	x	0	$1+x$	0	x
0	$1+x$	1	$1+x$	x	1	0	1	x	x	$1+x$	0
0	$1+x$	1	1	$1+x$	x	x	0	1	0	x	$1+x$
0	1	x	$1+x$	0	x	1	0	$1+x$	1	$1+x$	x
0	1	x	x	$1+x$	0	$1+x$	1	0	x	1	$1+x$
0	1	x	0	x	$1+x$	0	$1+x$	1	$1+x$	x	1
0	x	$1+x$	1	x	0	1	$1+x$	x	1	0	$1+x$
0	x	$1+x$	0	1	x	x	1	$1+x$	$1+x$	1	0
0	x	$1+x$	x	0	1	$1+x$	x	1	0	$1+x$	1